J. Fluid Mech. (1971), *vol.* 45, *part* 4, *pp.* 625–657
Printed in Great Britain

The motion of long slender bodies in a viscous fluid. Part 2. Shear flow

By R. G. COX

Pulp and Paper Research Institute of Canada and Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, Canada

(Received 15 April 1969)

A long slender axisymmetric body is considered placed at rest in a general linear flow in such a manner that the undisturbed fluid velocity is identically zero on the body axis. Formulae for the total force and torque on the body are found as an expansion in terms of a small parameter κ defined as the radius-to-length ratio of the body. These general results are used to determine the resistance to axial rotation of the body and also the equivalent axis ratio of the body for motion in a shear flow.

1. Introduction

The behaviour of a long slender solid body of circular cross-section in a given creeping motion flow has been considered by Cox (1970) and Tillet (1970), while Batchelor (1970) has examined the behaviour of such bodies of non-circular cross-section. Neglecting fluid inertia effects, Cox (1970) obtained the force per unit length acting on the body by the fluid as an asymptotic expansion in terms of a parameter κ defined as the ratio of the cross-sectional radius to body length. This theory, although successful in giving the translational resistance for such bodies, failed to give any results for cases in which the undisturbed flow field $\mathbf{U}(\mathbf{r})$ was identically zero on the body centre-line. One very important example in which this difficulty arises is concerned with the motion of a long thin axially symmetric solid body in shear flow.

Relative to a fixed system of axes, consider an undisturbed shear flow $\mathbf{U}(\mathbf{r})$ given by

$$\mathbf{U}(\mathbf{r}) = (0, 0, \gamma r_2). \tag{1.1}$$

Into this flow field an axially symmetric solid body with fore–aft symmetry is placed, the orientation of the body being determined by spherical polar angles θ and ϕ (see figure 1). If such a body is free to move, then, for the case of the body being an ellipsoid of revolution, it was shown by Jeffery (1922) that the motion of the body is periodic and given by

$$\tan\theta = \frac{Cr_e}{(r_e^2\cos^2\phi+\sin^2\phi)^{\frac{1}{2}}},$$
$$\tan\phi = r_e\tan(2\pi t/T), \tag{1.2}$$

where T is the period of the motion and has the value

$$T = (2\pi/\gamma)(r_e+r_e^{-1}), \tag{1.3}$$

the quantity r_e being the axis ratio of the ellipsoid. The constant C appearing in (1.2) is called the orbit constant and depends only upon the initial orientation of the body. Bretherton (1962) showed that the above formulae (1.2) and (1.3) were valid for general axisymmetric bodies with fore–aft symmetry, the constant r_e no longer being equal to the actual axis ratio of the body but being a function of the complete body shape. This constant r_e is therefore termed the 'equivalent axis ratio' of the body. The formulae (1.2) and (1.3) were verified experimentally for cylindrical bodies (rods and disks) by Trevelyan & Mason (1951), Mason & Manley (1956), Bartok & Mason (1957) and Goldsmith & Mason (1962) and for ellipsoidal bodies by Taylor (1923) and Anczurowski & Mason (1968).

From the equations (1.2) it may readily be shown that

$$\frac{d\phi}{dt} = \frac{\gamma}{r_e^2+1}(\sin^2\phi + r_e^2\cos^2\phi). \tag{1.4}$$

Consider a body moving in an orbit $C = \infty$ so that the motion is entirely in the r_2, r_3 plane ($\theta = \frac{1}{2}\pi$). Then, when the body axis is in the r_2 direction, the angular velocity is

$$\frac{d\phi}{dt} = \frac{\gamma r_e^2}{r_e^2+1}$$

and, when in the r_3 direction, the angular velocity is

$$\frac{d\phi}{dt} = \frac{\gamma}{r_e^2+1}.$$

Hence r_e^2 is the ratio of the angular velocity of the body when its axis is in the r_2 direction to that when its axis is in the r_3 direction. Now consider the body held firmly at rest with its axis in the r_2 direction. The fluid would then produce a couple on the body of magnitude G' say. Similarly for the body held firmly at rest with its axis in the r_3 direction, it would experience a couple of magnitude G'' say. The couples G' and G'' on the body must be proportional to its angular velocities if they were free to rotate in the above orientations. Thus one sees that the equivalent axis ratio r_e is given by

$$r_e = (G'/G'')^{\frac{1}{2}}. \tag{1.5}$$

The value of G' may be evaluated by using the results given by Cox (1970). However, G'' cannot be evaluated since, for the body at rest with its axis in the r_3 direction, the undisturbed flow field $\mathbf{U}(\mathbf{r})$ is identically zero on its centre-line.

In the present paper we therefore consider a long slender solid body of circular cross-section placed in a given creeping motion flow field $\mathbf{U}(\mathbf{r})$ which is identically zero on the body centre-line. For simplicity it is assumed that the body centre-line is straight and that the flow $\mathbf{U}(\mathbf{r})$ increases linearly with distance from an origin. The total force and torque acting on the body are found as an asymptotic expansion in terms of the body radius-to-length ratio κ. It is also shown how these results are modified by the presence of solid walls near the body considered.

In the final sections the general results are used to determine the rotational resistance and the equivalent axis ratio r_e for long slender axisymmetric bodies with fore–aft symmetry.

2. General problem

Consider a long slender body S of circular cross-section, the length of the body being $2a$ and a characteristic value of the cross-sectional radius being b ($b \ll l$). It is assumed that the body centre-line is straight so that one may take rectangular Cartesian axes with the 1 axis lying along the body centre-line, the origin of co-ordinates O lying at the mid-point between the ends of the body (see figure 1). This body is assumed to be placed in a fluid of viscosity μ. Then, by using dimensionless quantities based upon the length a, the viscosity μ and a characteristic velocity U, one defines a dimensionless position vector $\mathbf{r}$ relative to the co-ordinate system. The dimensionless cross-sectional radius of the body may be

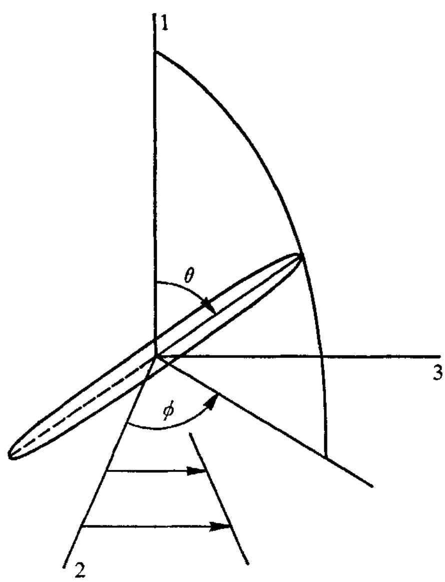

FIGURE 1. Spherical polar axes.

written as $\kappa\lambda(r_1)$, where κ is the ratio b/a and $\lambda(r_1)$ is a dimensionless function of $r_1 (-1 \leqslant r_1 \leqslant +1)$. The two ends of the body are then $r_1 = \pm 1$. It will be assumed that the shape of the body S determined by $\lambda(r_1)$ is such that (i) $\lambda(r_1)$ is a continuous function of r_1 and (ii) $\lambda(-1) = \lambda(+1) = 0$. Thus blunt-ended bodies such as a cylinder of finite length are omitted from the present theory. This is because it will be shown that for such cases the effect of flow around the body ends dominates over the effects of the flow around the rest of the body.

It is assumed that the fluid into which the body S is immersed is undergoing a motion $\mathbf{U}(\mathbf{r})$ which satisfies the creeping motion equations

$$\nabla^2\mathbf{U} - \nabla P = 0, \quad \nabla . \mathbf{U} = 0, \tag{2.1}$$

P being the dimensionless pressure field corresponding to $\mathbf{U}$. It is assumed that the body S is held fixed and that the flow field $\mathbf{U}(\mathbf{r})$ is identically zero along the r_1 axis. Also it is assumed that $\mathbf{U}(\mathbf{r})$ varies linearly with $\mathbf{r}$ so that

$$U_i = A_{ij} r_j \quad \text{and} \quad P = 0, \tag{2.2}$$

where A_{ij} is a constant second-order tensor. Now since $\mathbf{U} = 0$ for all r_1 if $r_2 = r_3 = 0$, it follows that

$$A_{i1} = 0 \quad \text{for all } i. \tag{2.3}$$

Hence

$$U_i = A_{i2}r_2 + A_{i3}r_3. \tag{2.4}$$

Changing to cylindrical polar axes (see figure 2) ρ, θ, z defined by

$$r_1 = z, \quad r_2 = \rho\cos\theta, \quad r_3 = \rho\sin\theta, \tag{2.5}$$

the components U_ρ, U_θ, U_z of $\mathbf{U}$ relative to these axes may be written in the form

$$\left.\begin{aligned} U_\rho &= U_2\cos\theta + U_3\sin\theta, \\ U_\theta &= U_3\cos\theta - U_2\sin\theta, \\ U_z &= U_1, \end{aligned}\right\} \tag{2.6}$$

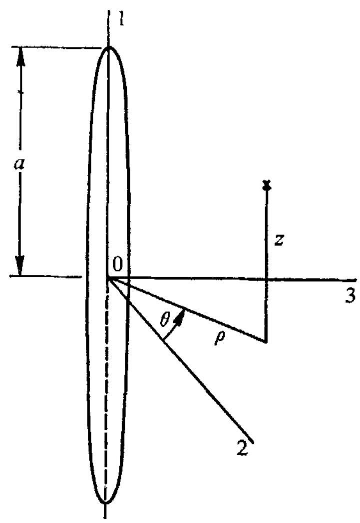

FIGURE 2. Cylindrical polar axes with the z-axis lying along body centre-line.

which, by making use of (2.4) and (2.5) and also by observing that $A_{22} + A_{33} = 0$ (from the equation of continuity $\nabla . \mathbf{U} = 0$), may be transformed into

$$\left.\begin{aligned} U_\rho &= A\rho\cos 2\theta + B\rho\sin 2\theta, \\ U_\theta &= -A\rho\sin 2\theta + B\rho\cos 2\theta + C\rho, \\ U_z &= D\rho\cos\theta + E\rho\sin\theta, \end{aligned}\right\} \tag{2.7}$$

where

$$\left.\begin{aligned} &A = A_{22}, \quad B = \tfrac{1}{2}(A_{23} + A_{32}), \\ &C = \tfrac{1}{2}(A_{32} - A_{23}), \\ &D = A_{12}, \quad E = A_{13}. \end{aligned}\right\} \tag{2.8}$$

The complete velocity field (i.e the flow field $\mathbf{U}$ together with the disturbance flow produced by the body S) is defined as $\mathbf{u}$, this flow field also satisfying the creeping motion equations

$$\nabla^2\mathbf{u} - \nabla p = 0, \quad \nabla . \mathbf{u} = 0, \tag{2.9}$$

p being the pressure field corresponding to $\mathbf{u}$.

One makes expansions of this flow field $(\mathbf{u}, p)$ in terms of the parameter

$$\kappa = b/a \tag{2.10}$$

in a manner similar to that used by Cox (1970) by defining an *outer* expansion in κ for which $\mathbf{r}$ is used as the independent variable and $\mathbf{u}$ and p as dependent variables. At each point P of the centre-line of the body S one may define an *inner* expansion in κ for which $\bar{\mathbf{r}}$ is used as the independent variable and $\bar{\mathbf{u}}$ and $\bar{p}$ as dependent variables, where $\bar{\mathbf{r}}$, $\bar{\mathbf{u}}$ and $\bar{p}$ are given by

$$\bar{\mathbf{r}} = (\mathbf{r}-\mathbf{r}^*)/\kappa, \quad \bar{\mathbf{u}} = \mathbf{u}, \quad \bar{p} = \kappa p, \tag{2.11}$$

where $\mathbf{r}^*$ is the position vector of the point P. In the outer expansion, a is the unit of length and, as $\kappa \to 0$, the body S becomes a line singularity (i.e. $b \to 0$) along the r_1 axis from $r_1 = -1$ to $r_1 = +1$, whereas, in the inner expansion at each point P of the centre-line, the unit of length is b and, as $\kappa \to 0$, the body S becomes very much like a cylinder of infinite length (since $a \to \infty$). Actually one has an infinite number of inner expansions corresponding to each point of the centre-line of the body S. However, all such inner expansions may be considered simultaneously by taking a general point P of the body centre-line. The inner expansion at such a point is then matched onto the solution for the outer expansion at the same point P.

3. Inner expansion

Consider the flow in the neighbourhood of a general point P on the centre-line of the body S. Since the undisturbed flow $\mathbf{U}(\mathbf{r})$ given by (2.7) is independent of z, it follows that, in inner variables for the inner expansion at P, this flow is given by

$$\left.\begin{aligned} U_\rho &= A\kappa\bar{\rho}\cos 2\theta + B\kappa\bar{\rho}\sin 2\theta, \\ U_\theta &= -A\kappa\bar{\rho}\sin 2\theta + B\kappa\bar{\rho}\cos 2\theta + C\kappa\bar{\rho}, \\ U_z &= D\kappa\bar{\rho}\cos\theta + E\kappa\bar{\rho}\sin\theta, \end{aligned}\right\} \tag{3.1}$$

the co-ordinates $(\bar{\rho}, \theta, \bar{z})$ being the polar co-ordinates of the inner expansion, i.e.

$$\bar{\rho} = \rho/\kappa, \quad \bar{z} = (z-z^*)/\kappa, \tag{3.2}$$

where z^* is the value of z at the point P.

Expressing (2.9) for the total flow field $(\mathbf{u}, p)$ in cylindrical polar co-ordinates (ρ, θ, z) and changing to inner variables $(\bar{\rho}, \theta, \bar{z})$, one obtains for the components $\bar{u}_\rho$, $\bar{u}_\theta$, $\bar{u}_z$ of the inner flow field,

$$\left.\begin{aligned} \frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}\left(\bar{\rho}\frac{\partial\bar{u}_\rho}{\partial\bar{\rho}}\right) + \frac{1}{\bar{\rho}^2}\frac{\partial^2\bar{u}_\rho}{\partial\theta^2} + \frac{\partial^2\bar{u}_\rho}{\partial\bar{z}^2} - \frac{2}{\bar{\rho}^2}\frac{\partial\bar{u}_\theta}{\partial\theta} - \frac{\bar{u}_\rho}{\bar{\rho}^2} - \frac{\partial\bar{p}}{\partial\bar{\rho}} &= 0, \\ \frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}\left(\bar{\rho}\frac{\partial\bar{u}_\theta}{\partial\bar{\rho}}\right) + \frac{1}{\bar{\rho}^2}\frac{\partial^2\bar{u}_\theta}{\partial\theta^2} + \frac{\partial^2\bar{u}_\theta}{\partial\bar{z}^2} + \frac{2}{\bar{\rho}^2}\frac{\partial\bar{u}_\rho}{\partial\theta} - \frac{\bar{u}_\theta}{\bar{\rho}^2} - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial\theta} &= 0, \\ \frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}\left(\bar{\rho}\frac{\partial\bar{u}_z}{\partial\bar{\rho}}\right) + \frac{1}{\bar{\rho}^2}\frac{\partial^2\bar{u}_z}{\partial\theta^2} + \frac{\partial^2\bar{u}_z}{\partial\bar{z}^2} - \frac{\partial\bar{p}}{\partial\bar{z}} &= 0, \\ \frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}(\bar{\rho}\bar{u}_\rho) + \frac{1}{\bar{\rho}}\frac{\partial\bar{u}_\theta}{\partial\theta} + \frac{\partial\bar{u}_z}{\partial\bar{z}} &= 0. \end{aligned}\right\} \tag{3.3}$$

Thus, writing $(\bar{\mathbf{u}}, \bar{p})$ in an expansion of the form

$$\left.\begin{aligned}\bar{\mathbf{u}} &= \kappa\bar{\mathbf{u}}_1 + \kappa^2\bar{\mathbf{u}}_2 + \ldots,\\ \bar{p} &= \kappa\bar{p}_1 + \kappa^2\bar{p}_2 + \ldots,\end{aligned}\right\} \tag{3.4}$$

it is seen that both the flow fields $(\bar{\mathbf{u}}_1, \bar{p}_1)$ and $(\bar{\mathbf{u}}_2, \bar{p}_2)$ satisfy equations of the form (3.3). Also, on the surface of the body S, one requires the no-slip boundary condition to apply, i.e.

$$\bar{\mathbf{u}} = 0 \quad \text{on} \quad S. \tag{3.5}$$

Now, in the neighbourhood of P, the surface of S may be written in outer variables as

$$\rho = \kappa[\lambda(z^*) + (z - z^*)\, d\lambda/dz^* + \ldots], \tag{3.6}$$

which, when expressed in inner variables, becomes

$$\bar{\rho} = \lambda(z^*) + \kappa\bar{z}\, d\lambda/dz^* + \ldots. \tag{3.7}$$

Therefore the inner boundary condition on $(\bar{\mathbf{u}}, \bar{p})$ becomes

$$\bar{\mathbf{u}} = 0 \quad \text{on} \quad \bar{\rho} = \lambda(z^*) + \kappa\bar{z}\, d\lambda/dz^* + \ldots. \tag{3.8}$$

One therefore lets

$$\bar{\mathbf{u}}_1 = 0 \quad \text{on} \quad \bar{\rho} = \lambda(z^*), \tag{3.9}$$

so that the value of $\bar{\mathbf{u}}_1$ on $\bar{\rho} = \lambda(z^*) + \kappa\bar{z}\,(d\lambda/dz^*) + \ldots$ is

$$(\bar{\mathbf{u}}_1)_S = \kappa\bar{z}\frac{d\lambda}{dz^*}\left(\frac{\partial\bar{\mathbf{u}}_1}{\partial\bar{\rho}}\right)_\lambda + \ldots, \tag{3.10}$$

where $(\partial\bar{\mathbf{u}}_1/\partial\bar{\rho})_\lambda$ is the value of $\partial\bar{\mathbf{u}}_1/\partial\bar{\rho}$ evaluated on $\bar{\rho} = \lambda(z^*)$. From the expansion (3.4), it is seen that, on $\bar{\rho} = \lambda(z^*) + \kappa\bar{z}(d\lambda/dz^*) + \ldots$, $\bar{\mathbf{u}}$ has the value

$$(\bar{\mathbf{u}}_1)_S = \kappa^2\bar{\mathbf{u}}_2 + \kappa^2\bar{z}\frac{d\lambda}{dz^*}\left(\frac{\partial\bar{\mathbf{u}}_1}{\partial\bar{\rho}}\right)_\lambda + \ldots, \tag{3.11}$$

so that the boundary condition (3.8) reduces to

$$\bar{\mathbf{u}}_2 = -\bar{z}\frac{d\lambda}{dz^*}\frac{\partial\bar{\mathbf{u}}_1}{\partial\bar{\rho}} \quad \text{on} \quad \bar{\rho} = \lambda(z^*). \tag{3.12}$$

In order to obtain the first-order flow field $(\bar{\mathbf{u}}_1, \bar{p}_1)$, one solves equations of the form (3.3) with inner boundary condition (3.9). From the form of equations (3.1), it is reasonable to assume an outer boundary condition for $(\bar{\mathbf{u}}_1, \bar{p}_1)$ of the form

$$\left.\begin{aligned}(\bar{u}_1)_\rho &\sim A\bar{\rho}\cos 2\theta + B\bar{\rho}\sin 2\theta,\\ (\bar{u}_1)_\theta &\sim A\bar{\rho}\sin 2\theta + B\bar{\rho}\cos 2\theta + C\bar{\rho},\\ (\bar{u}_1)_z &\sim +D\bar{\rho}\cos\theta + E\bar{\rho}\sin\theta \quad \text{as} \quad \bar{\rho}\to\infty.\end{aligned}\right\} \tag{3.13}$$

Therefore the flow field $(\bar{\mathbf{u}}_1, \bar{p}_1)$ is independent of $\bar{z}$ and so it may be shown that this flow field, in order to satisfy equations (3.3) and possess the asymptotic form (3.13), must be of the form

$$\left.\begin{aligned}(\bar{u}_1)_\rho &= \cos 2\theta[A\bar{\rho} + \alpha_1\bar{\rho}^{-1} + \alpha_2\bar{\rho}^{-3}] + \sin 2\theta[B\bar{\rho} + \alpha_3\bar{\rho}^{-1} + \alpha_4\bar{\rho}^{-3}],\\ (\bar{u}_1)_\theta &= \sin 2\theta[-A\bar{\rho} + \alpha_2\bar{\rho}^{-3}] + \cos 2\theta[B\bar{\rho} - \alpha_4\bar{\rho}^{-3}] + [C\bar{\rho} + \alpha_5\bar{\rho}^{-1}],\\ (\bar{u}_1)_z &= \cos\theta[D\bar{\rho} + \alpha_6\bar{\rho}^{-1}] + \sin\theta[E\bar{\rho} + \alpha_7\bar{\rho}^{-1}],\\ \bar{p}_1 &= \cos 2\theta[2\alpha_1\bar{\rho}^{-2}] + \sin 2\theta[2\alpha_3\bar{\rho}^{-2}],\end{aligned}\right\} \tag{3.14}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_7$ are constants which may be determined by applying the inner boundary condition (3.9). Thus the first-order inner flow field $(\bar{\mathbf{u}}_1, \bar{p}_1)$ may be written as

$$\left.\begin{aligned}(\bar{u}_1)_\rho &= A(\bar{\rho}-2\lambda^2\bar{\rho}^{-1}+\lambda^4\bar{\rho}^{-3})\cos 2\theta + B(\bar{\rho}-2\lambda^2\bar{\rho}^{-1}+\lambda^4\bar{\rho}^{-3})\sin 2\theta,\\ (\bar{u}_1)_\theta &= A(-\bar{\rho}+\lambda^4\bar{\rho}^{-3})\sin 2\theta + B(\bar{\rho}-\lambda^4\bar{\rho}^{-3})\cos 2\theta + C(\bar{\rho}-\lambda^2\bar{\rho}^{-1}),\\ (\bar{u}_1)_z &= D(\bar{\rho}-\lambda^2\bar{\rho}^{-1})\cos\theta + E(\bar{\rho}-\lambda^2\bar{\rho}^{-1})\sin\theta,\\ \bar{p}_1 &= -4A\lambda^2\bar{\rho}^{-2}\cos 2\theta - 4B\lambda^2\bar{\rho}^{-2}\sin 2\theta.\end{aligned}\right\} \quad (3.15)$$

On the surface $\bar{\rho} = \lambda$, the value of $\partial\bar{u}_1/\partial\bar{\rho}$ is therefore given by

$$\left.\begin{aligned}\partial(\bar{u}_1)_\rho/\partial\bar{\rho} &= 0,\\ \partial(\bar{u}_1)_\theta/\partial\bar{\rho} &= -4A\sin 2\theta + 4B\cos 2\theta + 2C,\\ \partial(\bar{u}_1)_z/\partial\bar{\rho} &= 2D\cos\theta + 2E\sin\theta,\end{aligned}\right\} \quad (3.16)$$

so that the boundary condition (3.12) on the second-order inner flow field $(\bar{\mathbf{u}}_2, \bar{p}_2)$ may be written as

$$\left.\begin{aligned}(\bar{u}_2)_\rho &= 0,\\ (\bar{u}_2)_\theta &= -\bar{z}(d\lambda/dz^*)(-4A\sin 2\theta + 4B\cos 2\theta + 2C),\\ (\bar{u}_2)_z &= -\bar{z}(d\lambda/dz^*)(2D\cos\theta + 2E\sin\theta) \quad \text{on} \quad \bar{\rho} = \lambda(z^*).\end{aligned}\right\} \quad (3.17)$$

Thus the flow field $(\bar{\mathbf{u}}_2, \bar{p}_2)$ satisfies the equations (3.3) with an inner boundary condition given by (3.17). However, no outer boundary condition will be imposed on this flow field at this stage since this will be determined by the matching of inner and outer expansions. Therefore one may write

$$\bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_2^* + \bar{\mathbf{u}}_2', \quad \bar{p}_2 = \bar{p}_2^* + \bar{p}_2', \quad (3.18)$$

where $(\bar{\mathbf{u}}_2^*, \bar{p}_2^*)$ is any particular flow field satisfying (3.3) with boundary conditions (3.17), so that the flow field $(\bar{\mathbf{u}}_2', \bar{p}_2')$ also satisfies (3.3) but satisfies the boundary condition

$$\bar{\mathbf{u}}_2' = 0 \quad \text{on} \quad \bar{\rho} = \lambda(z^*). \quad (3.19)$$

For the flow field $(\bar{\mathbf{u}}_2^*, \bar{p}_2^*)$, one may take

$$\begin{aligned}(\bar{u}_2^*)_\rho &= \bar{z}\cos 2\theta(\beta_1\bar{\rho}^{-1}+\beta_2\bar{\rho}^{-3}) + \bar{z}\sin 2\theta(\beta_4\bar{\rho}^{-1}+\beta_5\bar{\rho}^{-3}) + \cos\theta(\beta_8+\beta_{10}\bar{\rho}^{-2})\\ &\quad + \sin\theta(\beta_{11}+\beta_{13}\bar{\rho}^{-2}),\\ (\bar{u}_2^*)_\theta &= \bar{z}\sin 2\theta(\beta_2\bar{\rho}^{-3}) + \bar{z}\cos 2\theta(-\beta_5\bar{\rho}^{-3}) + \bar{z}(\beta_7\bar{\rho}^{-1}) + \sin\theta(\beta_9+\beta_{10}\bar{\rho}^{-2})\\ &\quad + \cos\theta(\beta_{12}-\beta_{13}\bar{\rho}^{-2}),\\ (\bar{u}_2^*)_z &= \cos 2\theta(-\tfrac{1}{2}\beta_1+\beta_3\bar{\rho}^{-2}) + \sin 2\theta(-\tfrac{1}{2}\beta_4+\beta_6\bar{\rho}^{-2}) + \bar{z}\cos\theta(-\beta_8-\beta_9)\bar{\rho}^{-1}\\ &\quad + \bar{z}\sin\theta(-\beta_{11}+\beta_{12})\bar{\rho}^{-1},\\ \bar{p}_2^* &= \bar{z}\cos 2\theta(2\beta_1\bar{\rho}^{-2}) + \bar{z}\sin 2\theta(2\beta_4\bar{\rho}^{-2}) + \cos\theta\{2(\beta_8+\beta_9)\bar{\rho}^{-1}\}\\ &\quad + \sin\theta\{2(\beta_{11}-\beta_{12})\bar{\rho}^{-1}\},\end{aligned} \quad (3.20)$$

which satisfies (3.3) identically for all values of the constants $\beta_1, \beta_2, \beta_3, \ldots, \beta_{13}$. Using the boundary conditions (3.17) in order to determine these constants, it is seen that the flow field $(\bar{\mathbf{u}}_2^*, \bar{p}_2^*)$ may be taken to be

$$\left.\begin{aligned}
(\bar{u}_2^*)_\rho &= \bar{z}\cos 2\theta(4A\lambda\, d\lambda/dz^*)(-\bar{\rho}^{-1}+\lambda^2\bar{\rho}^{-3})+\bar{z}\sin 2\theta(4B\lambda\, d\lambda/dz^*)\\
&\quad\times(-\bar{\rho}^{-1}+\lambda^2\bar{\rho}^{-3})+\cos\theta(D\lambda\, d\lambda/dz^*)(1-\lambda^2\bar{\rho}^{-2})\\
&\quad+\sin\theta(E\lambda\, d\lambda/dz^*)(1-\lambda^2\bar{\rho}^{-2}),\\
(\bar{u}_2^*)_\theta &= \bar{z}\sin 2\theta(4A\lambda\, d\lambda/dz^*)(\lambda^2\bar{\rho}^{-3})+\bar{z}\cos 2\theta(4B\lambda\, d\lambda/dz^*)(-\lambda^2\bar{\rho}^{-3})\\
&\quad+\bar{z}\bar{\rho}^{-1}(-2C\lambda\, d\lambda/dz^*)+\sin\theta(D\lambda\, d\lambda/dz^*)(1-\lambda^2\bar{\rho}^{-2})\\
&\quad+\cos\theta(E\lambda\, d\lambda/dz^*)(-1+\lambda^2\bar{\rho}^{-2}),\\
(\bar{u}_2^*)_z &= \cos 2\theta(2A\lambda\, d\lambda/dz^*)(1-\lambda^2\bar{\rho}^{-2})+\sin 2\theta(2B\lambda\, d\lambda/dz^*)(1-\lambda^2\bar{\rho}^{-2})\\
&\quad+\bar{z}\cos\theta(-2D\lambda\, d\lambda/dz^*)\bar{\rho}^{-1}+\bar{z}\sin\theta(-2E\lambda\, d\lambda/dz^*)\bar{\rho}^{-1},\\
\bar{p}_2^* &= \bar{z}\cos 2\theta(-8A\lambda\, d\lambda/dz^*)\bar{\rho}^{-2}+\bar{z}\sin 2\theta(-8B\lambda\, d\lambda/dz^*)\bar{\rho}^{-2}\\
&\quad+\cos\theta(4D\lambda\, d\lambda/dz^*)\bar{\rho}^{-1}+\sin\theta(4E\lambda\, d\lambda/dz^*)\bar{\rho}^{-1}.
\end{aligned}\right\} \quad (3.21)$$

[Note that this flow field $(\bar{\mathbf{u}}_2^*, \bar{p}_2^*)$ does not tend to zero as $\bar{\rho}\to\infty$.] By (3.4) and (3.18) the complete inner flow field $(\bar{\mathbf{u}}, \bar{p})$ is

$$\left.\begin{aligned}
\bar{\mathbf{u}} &= \kappa\bar{\mathbf{u}}_1+\kappa^2(\bar{\mathbf{u}}_2^*+\bar{\mathbf{u}}_2')+O(\kappa^3),\\
\bar{p} &= \kappa\bar{p}_1+\kappa^2(\bar{p}_2^*+\bar{p}_2')+O(\kappa^3),
\end{aligned}\right\} \quad (3.22)$$

where $(\bar{\mathbf{u}}_1, \bar{p}_1)$ and $(\bar{\mathbf{u}}_2^*, \bar{p}_2^*)$ are given respectively by the equations (3.15) and (3.21), the flow field $(\bar{\mathbf{u}}_2', \bar{p}_2')$ not yet being determined. Expressing the velocity field $(\kappa\bar{\mathbf{u}}_1+\kappa^2\bar{\mathbf{u}}_2^*)$ and pressure field $(\bar{p}_1+\kappa\bar{p}_2^*)$ in terms of outer variables, one obtains

$$\begin{aligned}
\kappa(\bar{u}_1)_\rho+\kappa^2(\bar{u}_2^*)_\rho &= \{A\rho\cos 2\theta+B\rho\sin 2\theta\}+\kappa^2\{-2A\lambda^2\rho^{-1}\cos 2\theta-2B\lambda^2\rho^{-1}\sin 2\theta\\
&\quad-(z-z^*)(4A\lambda\, d\lambda/dz^*)\rho^{-1}\cos 2\theta\\
&\quad-(z-z^*)(4B\lambda\, d\lambda/dz^*)\rho^{-1}\sin 2\theta\\
&\quad+(D\lambda\, d\lambda/dz^*)\cos\theta+(E\lambda\, d\lambda/dz^*)\sin\theta\}+O(\kappa^4),\\
\kappa(\bar{u}_1)_\theta+\kappa^2(\bar{u}_2^*)_\theta &= \{-A\rho\sin 2\theta+B\rho\cos 2\theta+C\rho\}\\
&\quad+\kappa^2\{-C\lambda^2\rho^{-1}-(2C\lambda\, d\lambda/dz^*)(z-z^*)\rho^{-1}\\
&\quad+(D\lambda\, d\lambda/dz^*)\sin\theta-(E\lambda\, d\lambda/dz^*)\cos\theta\}+O(\kappa^4),\\
\kappa(\bar{u}_1)_z+\kappa^2(\bar{u}_2^*)_z &= \{D\rho\cos\theta+E\rho\sin\theta\}\\
&\quad+\kappa^2\{-D\lambda^2\rho^{-1}\cos\theta-E\lambda^2\rho^{-1}\sin\theta+(2A\lambda\, d\lambda/dz^*)\cos 2\theta\\
&\quad+(2B\lambda\, d\lambda/dz^*)\sin 2\theta+(z-z^*)(-2D\lambda\, d\lambda/dz^*)\rho^{-1}\cos\theta\\
&\quad+(z-z^*)(-2E\lambda\, d\lambda/dz^*)\rho^{-1}\sin\theta\}+O(\kappa^4),\\
\bar{p}_1+\kappa\bar{p}_2^* &= \kappa^2\{-4A\lambda^2\rho^{-2}\cos 2\theta-4B\lambda^2\rho^{-2}\sin 2\theta\\
&\quad+(z-z^*)(-8A\lambda\, d\lambda/dz^*)\rho^{-2}\cos 2\theta\\
&\quad+(z-z^*)(-8B\lambda\, d\lambda/dz^*)\rho^{-2}\sin 2\theta+(4D\lambda\, d\lambda/dz^*)\rho^{-1}\cos\theta\\
&\quad+(4E\lambda\, d\lambda/dz^*)\rho^{-1}\sin\theta\}+O(\kappa^4).
\end{aligned} \quad (3.23)$$

In this equation (3.23), the quantity λ^2 is evaluated at $z = z^*$. Thus, by noting that

$$\lambda^2(z) = \lambda^2(z^*)+2(z-z^*)\lambda(z^*)\, d\lambda/dz^*+O(|z-z^*|^2),$$

(3.23) may be rewritten in the form

$$\left.\begin{aligned}
\kappa(\bar{u}_1)_\rho+\kappa^2(\bar{u}_2^*)_\rho &= \{A\rho\cos 2\theta+B\rho\sin 2\theta\}\\
&\quad+\kappa^2\{-2A\lambda^2\rho^{-1}\cos 2\theta-2B\lambda^2\rho^{-1}\sin 2\theta\\
&\quad+(D\lambda\, d\lambda/dz)\cos\theta+(E\lambda\, d\lambda/dz)\sin\theta\}+O(\kappa^4),\\
\kappa(\bar{u}_1)_\theta+\kappa^2(\bar{u}_2^*)_\theta &= \{-A\rho\sin 2\theta+B\rho\cos 2\theta+C\rho\}\\
&\quad+\kappa^2\{-C\lambda^2\rho^{-1}+(D\lambda\, d\lambda/dz)\sin\theta-(E\lambda\, d\lambda/dz)\cos\theta\}\\
&\quad+O(\kappa^4),\\
\kappa(\bar{u}_1)_z+\kappa^2(\bar{u}_2^*)_z &= \{D\rho\cos\theta+E\rho\sin\theta\}+\kappa^2\{-D\lambda^2\rho^{-1}\cos\theta\\
&\quad-E\lambda^2\rho^{-1}\sin\theta+(2A\lambda\, d\lambda/dz)\cos 2\theta\\
&\quad+(2B\lambda\, d\lambda/dz)\sin 2\theta\}+O(\kappa^4),\\
\bar{p}_1+\kappa\bar{p}_2^* &= \kappa^2\{-4A\lambda^2\rho^{-2}\cos 2\theta-4B\lambda^2\rho^{-2}\sin 2\theta\\
&\quad+(4D\lambda\, d\lambda/dz)\rho^{-1}\cos\theta+(4E\lambda\, d\lambda/dz)\rho^{-1}\sin\theta\}\\
&\quad+O(\kappa^4),
\end{aligned}\right\} \tag{3.24}$$

where the value of λ is now evaluated at z instead of z^*. In the terms of order κ^2 in the above equation (3.24), the expressions for the velocity are valid to $O(\rho^0)$ if $d\lambda/dz$ is also evaluated at z instead of z^*.

4. Outer expansion

For the outer expansion, the body S becomes a line singularity along the r_1 axis from $r_1 = -1$ to $r_1 = +1$ and near this singularity the outer flow field $(\mathbf{u}, p)$ must be matched onto the inner expansion. Since the velocity field $(\kappa\bar{\mathbf{u}}_1+\kappa^2\bar{\mathbf{u}}_2^*)$ and pressure field $(\bar{p}_1+\kappa\bar{p}_2^*)$ when expressed in outer variables is given by (3.24), it is reasonable to expect the outer velocity and pressure fields to be expandable in the form

$$\mathbf{u} = \mathbf{U}+\kappa^2\mathbf{u}_2+\ldots, \quad p = \kappa^2 p_2+\ldots, \tag{4.1}$$

where $\mathbf{U}(\mathbf{r})$ is the undisturbed velocity field given by (2.7). It is seen that the term of order κ^0 in (3.24) matches onto the term of order κ^0 in (4.1). Also comparing terms of order κ^2, it is seen that, as $\rho \to 0$, one needs

$$\left.\begin{aligned}
(u_2)_\rho &\sim (-2A\lambda^2\cos 2\theta-2B\lambda^2\sin 2\theta)\rho^{-1},\\
(u_2)_\theta &\sim (-C\lambda^2)\rho^{-1},\\
(u_2)_z &\sim (-D\lambda^2\cos\theta-E\lambda^2\sin\theta)\rho^{-1},\\
p_2 &\sim (-4A\lambda^2\cos 2\theta-4B\lambda^2\sin 2\theta)\rho^{-2}.
\end{aligned}\right\} \tag{4.2}$$

Since we require that $\mathbf{u} \sim \mathbf{U}$ as $\mathbf{r} \to \infty$, the outer boundary condition on $(\mathbf{u}_2, p_2)$ is that

$$\mathbf{u}_2 \to 0 \quad \text{as} \quad \mathbf{r} \to \infty. \tag{4.3}$$

Also the flow $(\mathbf{u}_2, p_2)$ must satisfy the creeping motion equations. It is therefore seen that the boundary conditions (4.2) and (4.3) can be satisfied by taking $(\mathbf{u}_2, p_2)$ to be a flow produced by a line of force doublets on the r_1 axis from $r_1 = -1$ to $r_1 = +1$. However, the creeping motion flow satisfying boundary conditions (4.2) and (4.3) is not unique since one could add on a line of force singularities on the r_1 axis from $r_1 = -1$ to $r_1 = +1$. Therefore one may write

$$\mathbf{u}_2 = \mathbf{u}_2^* + \mathbf{u}_2', \quad p_2 = p_2^* + p_2', \tag{4.4}$$

where $(\mathbf{u}_2', p_2')$ is a flow field satisfying the creeping motion equations and boundary condition (4.3) and has the property that, for $|r_1| \leqslant 1$,

$$\mathbf{u}_2' = o(\rho^{-1}) \quad \text{as} \quad \rho \to 0. \tag{4.5}$$

The flow field $(\mathbf{u}_2^*, p_2^*)$ then satisfies the boundary condition (4.2) and is the flow produced by a line of force doublets given by

$$\left.\begin{aligned}(u_2^*)_i &= \frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial r_k}\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i - R_i)(r_j - R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1,\\ p_2^* &= \frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial r_k}\left[\frac{2\delta_{ij}}{|\mathbf{r}-\mathbf{R}|^2}\right] dR_1,\end{aligned}\right\} \tag{4.6}$$

where $\mathbf{R}$ is the vector $(R_1, 0, 0)$ of a general point on the line singularity, and $g_{jk}(R_1)$ is the magnitude of the force doublet at $\mathbf{r} = \mathbf{R}$. We now examine the form of the velocity $\mathbf{u}_2^*$ near the line singularity. Thus consider points $\mathbf{r} = (r_1, r_2, r_3)$ with $|r_1| < 1$ and $r_2 = \rho\cos\theta$, $r_3 = \rho\sin\theta$ with $\rho \ll 1$. The integrals in (4.6) have a singularity in the integrand at $\mathbf{R} = \mathbf{r}$ if the point $\mathbf{r}$ actually lies on the singularity. Thus the range of integration $-1 \leqslant R_1 \leqslant 1$ is divided into the three separate intervals $-1 \leqslant R_1 \leqslant r_1 - \epsilon$, $r_1 - \epsilon \leqslant R_1 \leqslant r_1 + \epsilon$ and $r_1 + \epsilon \leqslant R_1 \leqslant +1$, where $\epsilon \ll 1$ is arbitrary and independent of ρ.

The contribution to the velocity field $\mathbf{u}_2^*$ from the force doublets $g_{12}(R_1)$ is given by

$$(u_2^*)_i = (1/8\pi)(I_i + J_i), \tag{4.7}$$

where

$$I_i = \int_{r_1-\epsilon}^{r_1+\epsilon} g_{12}(R_1)\frac{\partial}{\partial r_2}\left[\frac{\delta_{i1}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i - R_i)(r_1 - R_1)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1 \tag{4.8}$$

and

$$J_i = \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} g_{12}(R_1)\frac{\partial}{\partial r_2}\left[\frac{\delta_{i1}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i - R_i)(r_1 - R_1)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1. \tag{4.9}$$

The integral I_i may be evaluated by noting that R_1 is approximately equal to r_1 in the range of integration. Thus, by changing variables from R_1 to ϕ where

$$R_1 = r_1 + \rho\tan\phi, \tag{4.10}$$

and noting that one may write for R_1 in the range of integration

$$g_{12}(R_1) = g_{12}(r_1) + \rho\tan\phi\,\partial g_{12}/\partial r_1 + \ldots, \tag{4.11}$$

it is seen that I_i may therefore be written as

$$\begin{aligned}I_i = {} & \frac{g_{12}(r_1)}{\rho}\int_{-\tan^{-1}(\epsilon/\rho)}^{+\tan^{-1}(\epsilon/\rho)} \{\delta_{i1}(-\cos\theta\cos\phi - 3\cos\theta\sin^2\phi\cos\phi)\\ & + \delta_{i2}(-\sin\phi + 3\cos^2\theta\sin\phi\cos^2\phi) + 3\delta_{i3}\sin\theta\cos\theta\sin\phi\cos^2\phi\}\,d\phi\\ & + \frac{\partial g_{12}(r_1)}{\partial r_1}\int_{-\tan^{-1}(\epsilon/\rho)}^{+\tan^{-1}(\epsilon/\rho)}\bigg\{\delta_{i1}(-\cos\theta\sin\phi - 3\cos\theta\sin^3\phi)\\ & + \delta_{i2}\left(-\frac{\sin^2\phi}{\cos\phi} + 3\cos^2\theta\sin^2\phi\cos\phi\right) + 3\delta_{i3}\sin\theta\cos\theta\sin^2\phi\cos\phi\bigg\}\,d\phi\\ & + \ldots,\end{aligned}$$

which may be evaluated to give

$$I_i = -4g_{12}(r_1)\,\delta_{i1}\rho^{-1}\cos\theta + 2\frac{\partial g_{12}}{\partial r_1}\delta_{i2}\ln\rho + \frac{\partial g_{12}}{\partial r_1}\{\delta_{i2}(-\ln 4 - 2\ln\epsilon + 2 + 2\cos^2\theta) + 2\delta_{i3}\sin\theta\cos\theta\} + \dots \quad (4.12)$$

The integral J_i may be simplified by noting that there is no singularity in the range of integration even if $\mathbf{r}$ lies on the r_1 axis. Thus J_i must be of $O(\rho^0)$ as $\rho \to 0$ and may be evaluated to this order by simply putting $\rho = 0$ in the integrand. Thus

$$J_i = \delta_{i2}\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{g_{12}(R_1)\,(r_1-R_1)}{|r_1-R_1|^3}\,dR_1. \quad (4.13)$$

This expression is singular in ϵ as $\epsilon \to 0$ but this singularity may be shown to exactly cancel out the singularity of $-2\delta_{i2}(\partial g_{12}/\partial r_1)\ln\epsilon$ appearing in the expression (4.12) for I_i, so that the value of $\mathbf{u}_2^*$ tends to a finite limit as $\epsilon \to 0$. In a similar manner, the contributions of the force doublets g_{13}, g_{23}, g_{32}, g_{22} and g_{33} to I_i and J_i and hence to the flow field $\mathbf{u}_2^*$ may be found. Then assuming that g_{11}, g_{21} and g_{31} are zero (since their contribution to $\mathbf{u}_2^*$ may be transformed into that due to a line force distribution and so may be included in the flow field $\mathbf{u}_2'$), all the contributions to $\mathbf{u}_2^*$ may be added to give the asymptotic form of $\mathbf{u}_2^*$ near $\rho = 0$, $|r_1| < 1$ as being

$$\begin{aligned} 8\pi(u_2^*)_1 \sim\ & \rho^{-1}[-4g_{12}\cos\theta - 4g_{13}\sin\theta] \\ & + \ln\rho\left[2\frac{\partial g_{22}}{\partial r_1} + 2\frac{\partial g_{33}}{\partial r_1}\right] + \left[\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{(g_{22}+g_{33})\,(r_1-R_1)}{|r_1-R_1|^3}\,dR_1 \\ & + \left[2\frac{\partial g_{23}}{\partial r_1}\sin\theta\cos\theta + 2\frac{\partial g_{32}}{\partial r_1}\sin\theta\cos\theta\right. \\ & \left. + \frac{\partial g_{22}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\cos^2\theta) + \frac{\partial g_{33}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\sin^2\theta)\right], \end{aligned} \quad (4.14a)$$

$$\begin{aligned} 8\pi(u_2^*)_2 \sim\ & \rho^{-1}[2g_{23}(-\sin\theta - 2\sin\theta\cos^2\theta) + 2g_{32}(\sin\theta - 2\sin\theta\cos^2\theta) \\ & + 2g_{22}\cos\theta(1-2\cos^2\theta) + 2g_{33}\cos\theta(1-2\sin^2\theta)] \\ & + \ln\rho\left[2\frac{\partial g_{12}}{\partial r_1}\right] + \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{g_{12}(r_1-R_1)}{|r_1-R_1|^3}\,dR_1 \\ & + \left[\frac{\partial g_{12}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\cos^2\theta) + \frac{\partial g_{13}}{\partial r_1}\sin\theta\cos\theta\right], \end{aligned} \quad (4.14b)$$

$$\begin{aligned} 8\pi(u_2^*)_3 \sim\ & \rho^{-1}[2g_{23}(\cos\theta - 2\sin^2\theta\cos\theta) + 2g_{32}(-\cos\theta - 2\sin^2\theta\cos\theta) \\ & + 2g_{22}\sin\theta(1-2\cos^2\theta) + 2g_{33}\sin\theta(1-2\sin^2\theta)] \\ & + \ln\rho\left[2\frac{\partial g_{13}}{\partial r_1}\right] + \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{g_{13}(r_1-R_1)}{|r_1-R_1|^3}\,dR_1 \\ & + \left[\frac{\partial g_{13}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\sin^2\theta) + 2\frac{\partial g_{12}}{\partial r_1}\sin\theta\cos\theta\right]. \end{aligned} \quad (4.14c)$$

Thus, as $\rho \to 0$, the components $(u_2^*)_\rho$, $(u_2^*)_\theta$, $(u_2^*)_z$ of this velocity field have the asymptotic form

$$\begin{aligned}8\pi(u_2^*)_\rho \sim \rho^{-1}[&-4g_{23}\sin\theta\cos\theta-4g_{32}\sin\theta\cos\theta+2g_{22}(1-2\cos^2\theta)\\&+2g_{33}(1-2\sin^2\theta)]+\ln\rho\left[2\frac{\partial g_{12}}{\partial r_1}\cos\theta+2\frac{\partial g_{13}}{\partial r_1}\sin\theta\right]\\&+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(r_1-R_1)(g_{12}\cos\theta+g_{13}\sin\theta)}{|r_1-R_1|^3}dR_1\\&+\left[\left(\frac{\partial g_{12}}{\partial r_1}\cos\theta+\frac{\partial g_{13}}{\partial r_1}\sin\theta\right)(-\ln 4-2\ln\epsilon+2)+\left(2\frac{\partial g_{12}}{\partial r_1}\cos\theta+2\frac{\partial g_{13}}{\partial r_1}\sin\theta\right)\right],\end{aligned}\tag{4.15a}$$

$$\begin{aligned}8\pi(u_2^*)_\theta \sim \rho^{-1}[2g_{23}-2g_{32}]&+\ln\rho\left[-2\frac{\partial g_{12}}{\partial r_1}\sin\theta+2\frac{\partial g_{13}}{\partial r_1}\cos\theta\right]\\&+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(r_1-R_1)(-g_{12}\sin\theta+g_{13}\cos\theta)}{|r_1-R_1|^3}dR_1\\&+\left[\left(-\frac{\partial g_{12}}{\partial r_1}\sin\theta+\frac{\partial g_{13}}{\partial r_1}\cos\theta\right)(-\ln 4-2\ln\epsilon+2)\right],\end{aligned}\tag{4.15b}$$

$$\begin{aligned}8\pi(u_2^*)_z \sim \rho^{-1}[-4g_{12}\cos\theta-4g_{13}\sin\theta]&+\ln\rho\left[2\frac{\partial g_{22}}{\partial r_1}+2\frac{\partial g_{33}}{\partial r_1}\right]\\&+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(g_{22}+g_{33})(r_1-R_1)}{|r_1-R_1|^3}dR_1\\&+\left[2\left(\frac{\partial g_{23}}{\partial r_1}+\frac{\partial g_{32}}{\partial r_1}\right)\sin\theta\cos\theta+\frac{\partial g_{22}}{\partial r_1}(-\ln 4-2\ln\epsilon+2+2\cos^2\theta)\right.\\&\left.+\frac{\partial g_{33}}{\partial r_1}(-\ln 4-2\ln\epsilon+2+2\sin^2\theta)\right].\end{aligned}\tag{4.15c}$$

Since $\mathbf{u}_2^*$ must satisfy the boundary conditions (4.2) it follows that g_{12}, g_{13}, g_{23}, g_{32}, g_{22} and g_{33} must be chosen such that

$$\left.\begin{aligned}g_{23}+g_{32}&=B\lambda^2.8\pi,\\g_{22}-g_{33}&=A\lambda^2.8\pi,\\g_{23}-g_{32}&=-\tfrac{1}{2}C\lambda^2.8\pi,\\g_{12}&=\tfrac{1}{4}D\lambda^2.8\pi,\\g_{13}&=\tfrac{1}{4}E\lambda^2.8\pi.\end{aligned}\right\}\tag{4.16}$$

Now the terms

$$g_{22}\frac{\partial}{\partial r_2}\left\{\frac{\delta_{i2}}{|r-R|}+\frac{(r_i-R_i)r_2}{|r-R|^3}\right\}+g_{33}\frac{\partial}{\partial r_3}\left\{\frac{\delta_{i3}}{|r-R|}+\frac{(r_i-R_i)r_3}{|r-R|^3}\right\}$$

may be written in the form

$$\begin{aligned}\left(\frac{g_{22}-g_{33}}{2}\right)&\left[\frac{\partial}{\partial r_2}\left\{\frac{\delta_{i2}}{|r-R|}+\frac{(r_i-R_i)r_2}{|r-R|^3}\right\}-\frac{\partial}{\partial r_3}\left\{\frac{\delta_{i3}}{|r-R|}+\frac{(r_i-R_i)r_3}{|r-R|^3}\right\}\right]\\&+\left(\frac{g_{22}+g_{33}}{2}\right)\left[-\frac{\partial}{\partial r_1}\left\{\frac{\delta_{i1}}{|r-R|}+\frac{(r_i-R_i)(r_1-R_1)}{|r-R|^3}\right\}\right],\end{aligned}$$

where it is to be observed that the coefficient of $\frac{1}{2}(g_{22}+g_{33})$ is of the same form as that of g_{11} in the integrand of (4.6). Since it has already been assumed that $g_{11} = 0$, one may here take

$$g_{22}+g_{33} = 0. \tag{4.17}$$

Hence (4.26) and (4.27) give the values of the quantities $g_{12}, g_{13}, \ldots, g_{33}$ as being

$$\left.\begin{aligned} g_{12} = 2\pi D\lambda^2, \quad g_{13} = 2\pi E\lambda^2, \quad g_{23} = 2\pi(2B-C)\,\lambda^2, \\ g_{32} = 2\pi(2B+C)\,\lambda^2, \quad g_{22} = 4\pi A\lambda^2, \quad g_{33} = -4\pi A\lambda^2. \end{aligned}\right\} \tag{4.18}$$

Substituting these values into the asymptotic expansion (4.15) for $\mathbf{u}_2^*$ one obtains that as $\rho \to 0$

$$\begin{aligned}(u_2^*)_\rho \sim\ & \rho^{-1}(-2A\lambda^2\cos 2\theta - 2B\lambda^2 \sin 2\theta) + \ln\rho(\lambda\, d\lambda/dr_1)\,(D\cos\theta + E\sin\theta) \\ & + (\lambda\, d\lambda/dr_1)\,\{(D\cos\theta + E\sin\theta)\,(-\ln 2 - \ln\epsilon + 2)\} \\ & + \tfrac{1}{4}(D\cos\theta + E\sin\theta)\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda^2(r_1-R_1)}{|r_1-R_1|^3}\,dR_1, \end{aligned} \tag{4.19a}$$

$$\begin{aligned}(u_2^*)_\theta \sim\ & \rho^{-1}(-C\lambda^2) + \ln\rho(\lambda\, d\lambda/dr_1)\,(-D\sin\theta + E\cos\theta) \\ & + (\lambda\, d\lambda/dr_1)\,\{(-D\sin\theta + E\cos\theta)\,(-\ln 2 - \ln\epsilon + 1)\} \\ & + \tfrac{1}{4}(-D\sin\theta + E\cos\theta)\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda^2(r_1-R_1)}{|r_1-R_1|^3}\,dR_1, \end{aligned} \tag{4.19b}$$

$$(u_2^*)_z \sim \rho^{-1}(-D\lambda^2\cos\theta - E\lambda^2\sin\theta) + (\lambda\, d\lambda/dr_1)\,(2B\sin 2\theta + 2A\cos 2\theta). \tag{4.19c}$$

In order to obtain the velocity field $\mathbf{u}_2$, one has to add to $\mathbf{u}_2^*$ the velocity field $\mathbf{u}_2'$ (see equation (4.4)), where $\mathbf{u}_2'$ has the property (4.5). Thus $\mathbf{u}_2'$ may be taken to be the flow field produced by a line of forces $f_j(R_1)$ acting along the r_1 axis from $r_1 = -1$ to $r_1 = +1$. Therefore this velocity field is given by

$$(u_2')_i = \frac{1}{8\pi}\int_{-1}^{+1} f_j(R_1)\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i-R_i)\,(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1. \tag{4.20}$$

The asymptotic form of $\mathbf{u}_2'$ near the r_1 axis ($|r_1| < 1$) is now found in a manner similar to that for $\mathbf{u}_2^*$ by dividing the range of integration into the intervals $-1 \leqslant R_1 \leqslant r_1-\epsilon$, $r_1-\epsilon \leqslant R_1 \leqslant r_1+\epsilon$ and $r_1+\epsilon \leqslant R_1 \leqslant +1$. Thus writing

$$I_i = \int_{r_1-\epsilon}^{r_1+\epsilon} f_j(R_1)\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i-R_i)\,(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1$$

and

$$J_i = \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} f_j(R_1)\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i-R_i)\,(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1, \tag{4.21}$$

it may be shown that the contribution from $f_1(R_1)$ as $\rho \to 0$ is given by

$$\left.\begin{aligned} I_i &\sim 4f_1(R_1)\,\delta_{i1}(\ln 2 + \ln\epsilon - \ln\rho - \tfrac{1}{2}), \\ J_i &\sim 2\delta_{i1}\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{f_1(R_1)}{|r_1-R_1|}\,dR_1, \end{aligned}\right\} \tag{4.22}$$

the contributions from $f_2(R_1)$ and $f_3(R_1)$ to the asymptotic expansion of $\mathbf{u}_2'$ for $\rho \to 0$ being similarly obtained. The addition of these contributions then gives

$$8\pi(u_2') \sim 2(f_2\cos\theta+f_3\sin\theta)(\ln 2+\ln\epsilon-\ln\rho+1) + \left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_2\cos\theta+f_3\sin\theta)}{|r_1-R_1|}dR_1, \tag{4.23a}$$

$$8\pi(u_2')_\theta \sim 2(-f_2\sin\theta+f_3\cos\theta)(\ln 2+\ln\epsilon-\ln\rho) + \left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(-f_2\sin\theta+f_3\cos\theta)}{|r_1-R_1|}dR_1, \tag{4.23b}$$

$$8\pi(u_2')_z \sim 4f_1(\ln 2+\ln\epsilon-\ln\rho-\tfrac{1}{2})+2\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{f_1}{|r_1-R_1|}dR_1 \quad \text{as} \quad \rho\to 0. \tag{4.23c}$$

The asymptotic form of $\mathbf{u}_2$ as $\rho \to 0$ is obtained by adding the equations (4.19) and (4.23). Thus

$$\begin{aligned}(u_2)_\rho \sim{}& \rho^{-1}(-2A\lambda^2\cos 2\theta-2B\lambda^2\sin 2\theta)+\ln\rho\left\{\left(D\lambda\frac{d\lambda}{dr_1}-2\frac{f_2}{8\pi}\right)\cos\theta\right.\\ &\left.+\left(E\lambda\frac{d\lambda}{dr_1}-2\frac{f_3}{8\pi}\right)\sin\theta\right\}+\left\{\left(D\lambda\frac{d\lambda}{dr_1}-2\frac{f_2}{8\pi}\right)\cos\theta+\left(E\lambda\frac{d\lambda}{dr_1}-2\frac{f_3}{8\pi}\right)\sin\theta\right\}\\ &\times(-\ln 2-\ln\epsilon-1)+3\left(\lambda\frac{d\lambda}{dr_1}\right)(D\cos\theta+E\sin\theta)+\tfrac{1}{4}(+D\cos\theta+E\sin\theta)\\ &\times\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda^2(r_1-R_1)}{|r_1-R_1|^3}dR_1+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{f_2\cos\theta+f_3\sin\theta}{8\pi|r_1-R_1|}dR_1,\end{aligned} \tag{4.24a}$$

$$\begin{aligned}(u_2)_\theta \sim{}& \rho^{-1}(-C\lambda^2)+\ln\rho\left\{-\left(D\lambda\frac{d\lambda}{dr_1}-2\frac{f_2}{8\pi}\right)\sin\theta+\left(E\lambda\frac{d\lambda}{dr_1}-2\frac{f_3}{8\pi}\right)\cos\theta\right\}\\ &+\left\{-\left(D\lambda\frac{d\lambda}{dr_1}-2\frac{f_2}{8\pi}\right)\sin\theta+\left(E\lambda\frac{d\lambda}{dr_1}-2\frac{f_3}{8\pi}\right)\cos\theta\right\}(-\ln 2-\ln\epsilon)\\ &+\left(\lambda\frac{d\lambda}{dr_1}\right)(-D\sin\theta+E\cos\theta)+\tfrac{1}{4}(-D\sin\theta+E\cos\theta)\\ &\times\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda^2(r_1-R_1)}{|r_1-R_1|^3}dR_1+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(-f_2\sin\theta+f_3\cos\theta)}{8\pi|r_1-R_1|}dR_1,\end{aligned} \tag{4.24b}$$

$$\begin{aligned}(u_2)_z \sim{}& \rho^{-1}(-D\lambda^2\cos\theta-E\lambda^2\sin\theta)-4\frac{f_1}{8\pi}\ln\rho+4\frac{f_1}{8\pi}(\ln 2+\ln\epsilon-\tfrac{1}{2})\\ &+\left(\lambda\frac{d\lambda}{dr_1}\right)(2B\sin\theta+2A\cos 2\theta)+\frac{2}{8\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{f_1}{|r_1-R_1|}dR_1.\end{aligned} \tag{4.24c}$$

The value of $f_j(R_1)$ and hence of $\mathbf{u}_2$ may be a function of κ so long as the term $\kappa^2\mathbf{u}_2$ in (4.1) is larger than the next term (of order κ^3) in this expansion. It may be shown that one must take $f_j(R_1)$ to be of the form

$$f_j(R_1) = (f_0)_j+\frac{(f_1)_j}{\ln\kappa}+\frac{(f_2)_j}{(\ln\kappa)^2}+\dots, \tag{4.25}$$

so that $\mathbf{u}_2$ possesses an expansion

$$\mathbf{u}_2 = \mathbf{u}_2^*+\mathbf{u}_{20}+\frac{\mathbf{u}_{21}}{\ln\kappa}+\frac{\mathbf{u}_{22}}{(\ln\kappa)^2}+\dots, \tag{4.26}$$

where $\mathbf{u}_{20}$, $\mathbf{u}_{21}$ and $\mathbf{u}_{22}$ are given by (4.20) with $f_j(R_1)$ replaced by $(f_0)_j$, $(f_1)_j$ and $(f_2)_j$ respectively. Therefore the outer expansion (4.1) takes the form

$$\mathbf{u} = \mathbf{U}+\kappa^2(\mathbf{u}_2^*+\mathbf{u}_{20})+\frac{\kappa^2}{\ln\kappa}\mathbf{u}_{21}+\frac{\kappa^2}{(\ln\kappa)^2}\mathbf{u}_{22}+\ldots. \tag{4.27}$$

Substituting the expansion (4.25) into (4.24), one obtains the asymptotic forms of $\mathbf{u}_{20}$, $\mathbf{u}_{21}$, ... as $\rho\rightarrow 0$. By noting that the outer expansion can possess no term like $(\kappa^2\ln\rho)$ [since this would imply a term of order $(\kappa^2\ln\kappa)$ in the inner expansion], it follows from (4.24), (4.25) and (4.26) that $(f_0)_j$ is given by

$$(f_0)_1 = 0,\quad (f_0)_2 = 4\pi D\lambda\, d\lambda/dr_1,\quad (f_0)_3 = 4\pi E\lambda\, d\lambda/dr_1. \tag{4.28}$$

We now examine the integral

$$K = \left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda^2(r_1-R_1)}{|r_1-R_1|^3}dR \tag{4.29}$$

appearing in (4.24). Integrating this by parts, one obtains

$$K = \left[+\frac{\lambda^2}{|r_1-R_1|}\right]_{-1}^{r_1-\epsilon}+\left[+\frac{\lambda^2}{|r_1-R_1|}\right]_{r_1+\epsilon}^{+1}-2\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda\, d\lambda/dR_1}{|r_1-R_1|}dR_1. \tag{4.30}$$

Now, since
$$\lambda(-1) = \lambda(+1) = 0$$
and
$$\lambda^2(r_1-\epsilon) = \lambda^2(r_1)-2\lambda\, d\lambda/dr_1,$$
$$\lambda^2(r_1+\epsilon) = \lambda^2(r_1)+2\lambda\, d\lambda/dr_1,$$

it follows that the above expression (4.30) for K may be simplified to give

$$K = -4\lambda\frac{d\lambda}{dr_1}-2\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda\, d\lambda/dr_1}{|r_1-R_1|}dR_1. \tag{4.31}$$

By substituting the expansion (4.25) into (4.24) and by making use of the results (4.28) and (4.31), one may obtain the asymptotic form of $\mathbf{u}_2^*+\mathbf{u}_{20}$ for $\rho\rightarrow 0$ as

$$\left.\begin{aligned}(u_2^*)_\rho+(u_{20})_\rho &\sim \rho^{-1}(-2A\lambda^2\cos 2\theta-2B\lambda^2\sin 2\theta)+2(\lambda\, d\lambda/dr_1)(D\cos\theta+E\sin\theta),\\ (u_2^*)_\theta+(u_{20})_\theta &\sim \rho^{-1}(-C\lambda^2),\\ (u_2^*)_z+(u_{20})_z &\sim \rho^{-1}(-D\lambda^2\cos\theta-E\lambda^2\sin\theta)+(\lambda\, d\lambda/dr_1)(2B\sin 2\theta+2A\cos 2\theta).\end{aligned}\right\} \tag{4.32}$$

For the inner expansion, the velocity field $\bar{\mathbf{u}}$ is given by

$$\bar{\mathbf{u}} = \kappa\bar{\mathbf{u}}_1+\kappa^2(\bar{\mathbf{u}}_2^*+\bar{\mathbf{u}}_2')+O(\kappa^3), \tag{4.33}$$

where $(\kappa\bar{\mathbf{u}}_1+\kappa^2\bar{\mathbf{u}}_2^*)$, when expressed in outer variables, is of the form (3.24). The terms of order κ^0 in (3.24) have already been matched and it is seen that in order that terms of order κ^2 be matched one has to choose a velocity field $\bar{\mathbf{u}}_2'$ in the inner expansion such that when expressed in outer variables

$$\left.\begin{aligned}(\bar{u}_2')_\rho &\sim (\lambda\, d\lambda/dr_1)(D\cos\theta+E\sin\theta),\\ (\bar{u}_2')_\theta &\sim (\lambda\, d\lambda/dr_1)(-D\sin\theta+E\cos\theta),\\ (\bar{u}_2')_z &\sim 0,\quad \text{as}\quad \rho\rightarrow 0.\end{aligned}\right\} \tag{4.34}$$

Thus, relative to the Cartesian axes,

$$\left.\begin{aligned}&(\bar{u}_2')_1\rightarrow 0,\quad (\bar{u}_2')_2\rightarrow(\lambda\, d\lambda/dr_1)D,\\ &(\bar{u}_2')_3\rightarrow(\lambda\, d\lambda/dr_1)E\quad \text{as}\quad \rho\rightarrow 0,\end{aligned}\right\} \tag{4.35}$$

which represents a uniform flow. Also $\bar{\mathbf{u}}_2'$ satisfies the creeping motion equations and the no-slip boundary condition (3.19) on $\bar{\rho} = \lambda(r_1)$. Hence one may find the flow field $\bar{\mathbf{u}}_2'$ by following the analysis given by Cox (1970). Thus we take

$$\left.\begin{aligned}(\bar{u}_2')_\rho &= H\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta+K\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\sin\theta,\\ (\bar{u}_2')_\theta &= H\{1-\lambda^2\bar{\rho}^{-2}+2\ln(\bar{\rho}/\lambda)\}\sin\theta+K\{-1+\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta,\\ (\bar{u}_2')_z &= 0,\end{aligned}\right\} \quad (4.36)$$

where H and K may be functions of κ so long as they are much larger than κ in the limit of $\kappa \to 0$. We shall take

$$\left.\begin{aligned}H &= \frac{H_1}{\ln\kappa}+\frac{H_2}{(\ln\kappa)^2}+\dots,\\ K &= \frac{K_1}{\ln\kappa}+\frac{K_2}{(\ln\kappa)^2}+\dots.\end{aligned}\right\} \quad (4.37)$$

The substitution of (4.37) into (4.36) and the changing to outer variables yields

$$\left.\begin{aligned}(\bar{u}_2')_\rho &= (2H_1\cos\theta+2K_1\sin\theta)+\frac{1}{\ln\kappa}\{(2H_2+H_1-2H_1\ln(\rho/\lambda))\cos\theta\\ &\quad+(2K_2+K_1-2K_1\ln(\rho/\lambda))\sin\theta\}+\dots,\\ (\bar{u}_2')_\theta &= (-2H_1\sin\theta+2K_1\cos\theta)+\frac{1}{\ln\kappa}\{(-2H_2+H_1+2H_1\ln(\rho/\lambda))\sin\theta\\ &\quad+(2K_2-K_1-2K_1\ln(\rho/\lambda))\cos\theta\}+\dots,\\ (\bar{u}_2')_z &= 0.\end{aligned}\right\} \quad (4.38)$$

Hence by the condition (4.34), it is seen that H_1 and K_1 are given by

$$H_1 = \tfrac{1}{2}\lambda\,(d\lambda/dr_1)\,D, \quad K_1 = \tfrac{1}{2}\lambda\,(d\lambda/dr_1)\,E. \quad (4.39)$$

In the outer expansion, the substitution of (4.25) and (4.26) into (4.24) and the equating terms of order $(\ln\kappa)^{-1}$ give the asymptotic form of $\mathbf{u}_{21}$ for $\rho \to 0$ as being

$$\begin{aligned}8\pi(u_{21})_\rho &\sim \ln\rho\{-2(f_1)_2\cos\theta-2(f_1)_3\sin\theta\}+\{-2(f_1)_2\cos\theta-2(f_1)_3\sin\theta\}\\ &\quad\times(-\ln 2-\ln\epsilon-1)+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_2\cos\theta+(f_1)_3\sin\theta}{|r_1-R_1|}\,dR_1,\end{aligned} \quad (4.40a)$$

$$\begin{aligned}8\pi(u_{21})_\theta &\sim \ln\rho\{+2(f_1)_2\sin\theta-2(f_1)_3\cos\theta\}+\{+2(f_1)_2\sin\theta-2(f_1)_3\cos\theta\}\\ &\quad\times(-2\ln 2-\ln\epsilon)+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{-(f_1)_2\sin\theta+(f_1)_3\cos\theta}{|r_1-R_1|}\,dR_1,\end{aligned} \quad (4.40b)$$

$$8\pi(u_{21})_z \sim -4(f_1)_1\ln\rho+4(f_1)_1(\ln 2+\ln\epsilon-\tfrac{1}{2})+2\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_1}{|r_1-R_1|}\,dR_1. \quad (4.40c)$$

The matching of terms involving $\ln\rho$ in (4.38) and (4.40) gives

$$(f_1)_1 = 0, \quad (f_1)_2 = 8\pi H_1, \quad (f_1)_3 = 8\pi K_1,$$

so that by (4.39)

$$(f_1)_2 = 4\pi\lambda(d\lambda/dr_1)\,D, \quad (f_1)_3 = 4\pi\lambda(d\lambda/dr_1)\,E. \quad (4.41)$$

Also the matching of terms of order ρ^0 in (4.38) and (4.40) gives the values of H_2 and K_2 as

$$\left.\begin{aligned} H_2 &= H_1(\tfrac{1}{2}-\ln\lambda)+\frac{(f_1)_2}{8\pi}(\ln 2+\ln\epsilon)+\frac{1}{16\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_2}{|r_1-R_1|}dR_1,\\ K_2 &= K_1(\tfrac{1}{2}-\ln\lambda)+\frac{(f_1)_3}{8\pi}(\ln 2+\ln\epsilon)+\frac{1}{16\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_3}{|r_1-R_1|}dR_1.\end{aligned}\right\} \quad (4.42)$$

The substitution into these expressions (4.42) of the values of $H_1, K_1, (f_1)_2$ and $(f_1)_3$ from (4.39) and (4.41) yields

$$\left.\begin{aligned} H_2 &= \tfrac{1}{2}\lambda(d\lambda/dr_1)D[\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon]+\tfrac{1}{4}D\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)}{|r_1-R_1|}dR_1,\\ K_2 &= \tfrac{1}{2}\lambda(d\lambda/dr_1)E[\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon]+\tfrac{1}{4}E\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)}{|r_1-R_1|}dR_1.\end{aligned}\right\} \quad (4.43)$$

Continuing the matching process by substituting (4.25) and (4.26) into (4.24) and equating terms of order $(\ln\kappa)^{-2}$ give the asymptotic form of $\mathbf{u}_{22}$ for $\rho\to 0$. Then, by matching the term of order $(\ln\rho)$ with that appearing in the coefficient of $(\ln\kappa)^{-2}$ in (4.39), one obtains the values of $(f_2)_i$ as

$$(f_2)_1 = 0, \quad (f_2)_2 = 8\pi H_2, \quad (f_2)_3 = 8\pi K_2,$$

so that

$$(f_2)_2/8\pi = \tfrac{1}{2}\lambda(d\lambda/dr_1)D[\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon]+\tfrac{1}{4}D\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)}{|r_1-R_1|}dR_1, \quad (4.44a)$$

$$(f_2)_3/8\pi = \tfrac{1}{2}\lambda(d\lambda/dr_1)E[\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon]+\tfrac{1}{4}E\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)}{|r_1-R_1|}dR_1. \quad (4.44b)$$

5. Force on body

The results given in §§3 and 4 for the velocity field are now collected. The equations (3.4), (3.15), (3.18), (3.21), (4.36), (4.37), (4.39) and (4.43) give the inner velocity field $\bar{\mathbf{u}}$ as being

$$\bar{\mathbf{u}} = \kappa\bar{\mathbf{u}}_1+\kappa^2\left\{\bar{\mathbf{u}}_2^*+\frac{\bar{\mathbf{u}}'_{21}}{\ln\kappa}+\frac{\bar{\mathbf{u}}'_{22}}{(\ln\kappa)^2}+\ldots\right\}+\ldots, \quad (5.1)$$

where $\bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_2^*$, $\bar{\mathbf{u}}'_{21}$ and $\bar{\mathbf{u}}'_{22}$ are given by

$$\left.\begin{aligned} (\bar{u}_1)_\rho &= A(\bar{\rho}-2\lambda^2\bar{\rho}^{-1}+\lambda^4\bar{\rho}^{-3})\cos 2\theta+B(\bar{\rho}-2\lambda^2\bar{\rho}^{-1}+\lambda^4\bar{\rho}^{-3})\sin 2\theta,\\ (\bar{u}_1)_\theta &= A(-\bar{\rho}+\lambda^4\bar{\rho}^{-3})\sin 2\theta+B(\bar{\rho}-\lambda^4\bar{\rho}^{-3})\cos 2\theta+C(\bar{\rho}-\lambda^2\bar{\rho}^{-1}),\\ (\bar{u}_1)_z &= D(-\bar{\rho}+\lambda^2\bar{\rho}^{-1})\cos\theta+E(\bar{\rho}-\lambda^2\bar{\rho}^{-1})\sin\theta;\end{aligned}\right\} \quad (5.2)$$

$$\left.\begin{aligned} (\bar{u}_2^*)_\rho &= 4\lambda(d\lambda/dr_1)(-\bar{\rho}^{-1}+\lambda^2\bar{\rho}^{-3})\bar{z}(A\cos 2\theta+B\sin 2\theta)\\ &\quad +\lambda(d\lambda/dr_1)(1-\lambda^2\bar{\rho}^{-2})(D\cos\theta+E\sin\theta),\\ (\bar{u}_2^*)_\theta &= 4\lambda^3(d\lambda/dr_1)\bar{\rho}^{-3}\bar{z}(A\sin 2\theta-B\cos 2\theta)\\ &\quad -2C\lambda(d\lambda/dr_1)\bar{\rho}^{-1}\bar{z}+\lambda(d\lambda/dr_1)(1-\lambda^2\bar{\rho}^{-2})(D\sin\theta-E\cos\theta),\\ (\bar{u}_2^*)_z &= 2\lambda(d\lambda/dr_1)(1-\lambda^2\bar{\rho}^{-2})(A\cos 2\theta+B\sin 2\theta)\\ &\quad -2\lambda(d\lambda/dr_1)\bar{\rho}^{-1}\bar{z}(D\cos\theta+E\sin\theta);\end{aligned}\right\} \quad (5.3)$$

$$\left.\begin{aligned}(\bar{u}'_{21})_\rho &= \tfrac{1}{2}\lambda(d\lambda/dr_1)\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}D\cos\theta\\ &\quad+\tfrac{1}{2}\lambda(d\lambda/dr_1)\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}E\sin\theta,\\ (\bar{u}'_{21})_\theta &= \tfrac{1}{2}\lambda(d\lambda/dr_1)\{1-\lambda^2\bar{\rho}^{-2}+2\ln(\bar{\rho}/\lambda)\}D\sin\theta\\ &\quad+\tfrac{1}{2}\lambda(d\lambda/dr_1)\{-1+\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}E\cos\theta,\\ (\bar{u}'_{21})_z &= 0;\end{aligned}\right\} \tag{5.4}$$

$$\left.\begin{aligned}(\bar{u}'_{22})_\rho &= H_2\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta+K_2\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\sin\theta,\\ (\bar{u}'_{22})_\theta &= H_2\{1-\lambda^2\bar{\rho}^{-2}+2\ln(\bar{\rho}/\lambda)\}\sin\theta+K_2\{-1+\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta,\\ (\bar{u}'_{22})_z &= 0;\end{aligned}\right\} \tag{5.5}$$

where H_2 and K_2 are constants given by

$$H_2 = \tfrac{1}{2}\lambda(d\lambda/dr_1)\,D\{\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon\}+\tfrac{1}{4}D\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda d\lambda/dR_1}{|r_1-R_1|}dR_1,$$

$$K_2 = \tfrac{1}{2}\lambda(d\lambda/dr_1)\,E\{\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon\}+\tfrac{1}{4}E\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda\, d\lambda/dR_1}{|r_1-R_1|}dR_1. \tag{5.6}$$

Also, from (2.7), (4.27), (4.6), (4.18), (4.20), (4.28), (4.41) and (4.44), it is seen that the velocity field $\mathbf{u}$ in the outer expansion may be written as

$$\mathbf{u} = \mathbf{U}+\kappa^2\left\{(\mathbf{u}_2^*+\mathbf{u}_{20})+\frac{\mathbf{u}_{21}}{\ln\kappa}+\frac{\mathbf{u}_{22}}{(\ln\kappa)^2}+\dots\right\}, \tag{5.7}$$

where $\mathbf{U}$ is the undisturbed flow field

$$\left.\begin{aligned}U_\rho &= A\rho\cos 2\theta+B\rho\sin 2\theta,\\ U_\theta &= -A\rho\sin 2\theta+B\rho\cos 2\theta+C\rho,\\ U_z &= D\rho\cos\theta+E\rho\sin\theta;\end{aligned}\right\} \tag{5.8}$$

and where $\mathbf{u}_2^*$ is the flow

$$(u_2^*)_i = \frac{1}{8\pi}\int_{-1}^{+1}g_{jk}(R_1)\frac{\partial}{\partial r_k}\left[\frac{\delta_{ij}}{|r-R|}+\frac{(r_i-R_i)(r_j-R_j)}{|r-R|^3}\right]dR_1 \tag{5.9}$$

produced by a line of force doublets of magnitude $g_{jk}(R_1)$ given by the matrix

$$g_{jk}(R_1) = 2\pi\lambda^2\begin{pmatrix}0 & D & E\\ 0 & 2A & (2B-C)\\ 0 & (2B+C) & -2A\end{pmatrix}. \tag{5.10}$$

The remaining flow field $\kappa^2\{\mathbf{u}_{20}+(\mathbf{u}_{21}/\ln\kappa)+(\mathbf{u}_{22}/(\ln\kappa)^2)\}$ in the equation (5.7) is given by

$$\left\{(u_{20})_i+\frac{(u_{21})_i}{\ln\kappa}+\frac{(u_{22})_i}{(\ln\kappa)^2}\right\} = \frac{1}{8\pi}\int_{-1}^{+1}\left\{(f_0)_j+\frac{(f_1)_j}{\ln\kappa}+\frac{(f_2)_j}{(\ln\kappa)^2}\right\}\times\left[\frac{\delta_{ij}}{|r-R|}+\frac{(r_i-R_i)(r_j-R_j)}{|r-R|^3}\right]dR_1, \tag{5.11}$$

which is a flow produced by a line of force of magnitude

$$\{(f_0)_j+(f_1)_j/\ln\kappa+(f_2)_j/(\ln\kappa)^2\},$$

where $(f_0)_j$, $(f_1)_j$ and $(f_2)_j$ are given by

$$\mathbf{f}_0 = \mathbf{f}_1 = 4\pi\lambda \frac{d\lambda}{dr_1}\begin{pmatrix} 0 \\ D \\ E \end{pmatrix}, \tag{5.12}$$

$$\mathbf{f}_2 = 8\pi \begin{pmatrix} 0 \\ H_2 \\ K_2 \end{pmatrix}, \tag{5.13}$$

the quantities H_2 and K_2 being given by (5.6). Hence from the outer expansion it is seen that the forces acting on the body are equivalent to a force $\mathscr{F}$ and a couple $\mathscr{G}$ per unit length acting on it, where

$$\mathscr{F} = 8\pi\kappa^2 \begin{pmatrix} 0 \\ -\tfrac{1}{2}\lambda \dfrac{d\lambda}{dr_1} D\left(1 + \dfrac{1}{\ln\kappa}\right) - \dfrac{H_2}{(\ln\kappa)^2} \\ -\tfrac{1}{2}\lambda \dfrac{d\lambda}{dr_1} E\left(1 + \dfrac{1}{\ln\kappa}\right) - \dfrac{K_2}{(\ln\kappa)^2} \end{pmatrix},$$

$$\mathscr{G} = \kappa^2 \begin{pmatrix} g_{32} - g_{23} \\ g_{13} \\ -g_{12} \end{pmatrix} = 2\pi\lambda^2\kappa^2 \begin{pmatrix} 2C \\ E \\ -D \end{pmatrix}. \tag{5.14}$$

The total force $\mathbf{F}$ and torque $\mathbf{G}$ acting on the body about the origin are given by

$$\mathbf{F} = \int_{-1}^{+1} \mathscr{F} dr_1, \quad \mathbf{G} = \int_{-1}^{+1} (\mathscr{G} + \mathbf{r} \times \mathscr{F})\, dr_1, \tag{5.15}$$

where $\mathbf{r}$ is the position vector of a general point on the body centre-line and is therefore of the form $(r_1, 0, 0)$. Since $\lambda = 0$ at $r = \pm 1$ it may readily be shown that

$$\int_{-1}^{+1} \lambda \frac{d\lambda}{dr_1} dr_1 = 0 \tag{5.16}$$

and

$$\int_{-1}^{+1} r_1 \lambda \frac{d\lambda}{dr_1} dr_1 = -\frac{1}{2}\int_{-1}^{+1} \lambda^2 dr_1. \tag{5.17}$$

Thus, substituting the expressions (5.14) into (5.15) and by making use of (5.16) and (5.17), one obtains for the total force $\mathbf{F}$ and torque $\mathbf{G}$ acting on the body

$$\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \tag{5.18}$$

where

$$\left.\begin{aligned} F_1 = 0, \quad F_2/8\pi &= -\frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} H_2\, dr_1, \\ F_3/8\pi &= -\frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} K_2\, dr_1, \end{aligned}\right\} \tag{5.19}$$

and

$$\left.\begin{aligned} G_1/8\pi &= \tfrac{1}{2}\kappa^2 C\int_{-1}^{+1}\lambda^2 dr_1,\\ G_2/8\pi &= -\frac{1}{4}\frac{\kappa^2}{\ln\kappa}E\int_{-1}^{+1}\lambda^2 dr_1+\frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1}r_1K_2dr_1,\\ G_3/8\pi &= +\frac{1}{4}\frac{\kappa^2}{\ln\kappa}D\int_{-1}^{+1}\lambda^2 dr_1-\frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1}r_1H_2dr_1.\end{aligned}\right\} \tag{5.20}$$

It should be noted that the terms of order κ^2 in the expressions for G_2 and G_3 are identically zero. If one substitutes the values of H_2 and K_2 from (5.6) into the formulae (5.19) and (5.20) and notes that in addition to (5.16) and (5.17) one has

$$\int_{-1}^{+1}\lambda\ln\lambda\frac{d\lambda}{dr_1}dr_1 = 0,$$

$$\int_{-1}^{+1}r_1\lambda\ln\lambda\frac{d\lambda}{dr_1}dr_1 = -\frac{1}{2}\int_{-1}^{+1}\lambda^2\ln\lambda\, dr_1+\frac{1}{4}\int_{-1}^{+1}\lambda^2 dr_1, \tag{5.21}$$

then the above values of components of $\mathbf{F}$ and $\mathbf{G}$ may be written as

$$\left.\begin{aligned} F_2/8\pi &= -\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}D\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)\,dR_1}{|r_1-R_1|}\right]dr_1,\\ F_3/8\pi &= -\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}E\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda(d\lambda/dR_1)\,dR_1}{|r_1-R_1|}\right]dr_1,\end{aligned}\right\} \tag{5.22}$$

and

$$\left.\begin{aligned} G_1/8\pi &= \tfrac{1}{2}\kappa^2 C\int_{-1}^{+1}\lambda^2 dr_1,\\ G_2/8\pi &= -\frac{\kappa^2}{\ln\kappa}\tfrac{1}{4}E\int_{-1}^{+1}\lambda^2 dr_1-\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}E\left\{(1+\ln 2+\ln\epsilon)\int_{-1}^{+1}\lambda^2 dr_1-\int_{-1}^{+1}\lambda^2\ln\lambda\, dr_1\right\}\\ &\quad+\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}E\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda\,(d\lambda/dR_1)\,dR_1}{|r_1-R_1|}\right]r_1dr_1,\\ G_3/8\pi &= +\frac{\kappa^2}{\ln\kappa}\tfrac{1}{4}D\int_{-1}^{+1}\lambda^2 dr_1+\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}D\left\{(1+\ln 2+\ln\epsilon)\int_{-1}^{+1}\lambda^2 dr_1-\int_{-1}^{+1}\lambda^2\ln\lambda\, dr_1\right\}\\ &\quad-\frac{\kappa^2}{(\ln\kappa)^2}\tfrac{1}{4}D\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{\lambda\,(d\lambda/dR_1)\,dR_1}{|r_1-R_1|}\right]r_1dr_1.\end{aligned}\right\} \tag{5.23}$$

6. Wall effects

The theory given in §§ 3, 4 and 5 for the hydrodynamic force and torque exerted on a long slender body S is valid for an undisturbed fluid flow of the form (2.7), there being no solid boundaries present. We now consider the possibility of a solid wall W being present at a distance from the body of order a. It is assumed that upon W the fluid has a given velocity $\mathbf{U}_W$. Thus one might for example consider (i) a body S placed in a shear flow bounded by a plane solid wall W, for which the undisturbed flow $\mathbf{U}$ (relative to axes fixed in the body) is given by (see figure 3)

$$U_i = D\delta_{i1}r_3,$$

which possesses the value $\quad (U_W)_i = -D\delta_{i1}k$

on the wall W at $r_3 = -k$; (ii) a body S rotating about its r_1 axis in a fluid at rest bounded by a plane wall W at $r_2 = -k$ (see figure 4). By adding a rotational flow to the system about the r_1 axis it may be shown that this problem is equivalent to one for which

$$U_1 = 0, \quad U_2 = -Cr_3, \quad U_3 = +Cr_2,$$

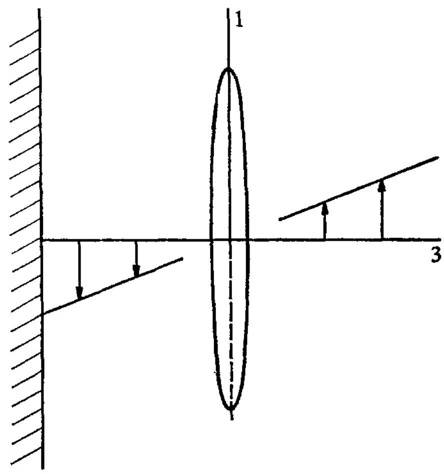

FIGURE 3. Body in shear flow in the neighbourhood of a plane solid wall.

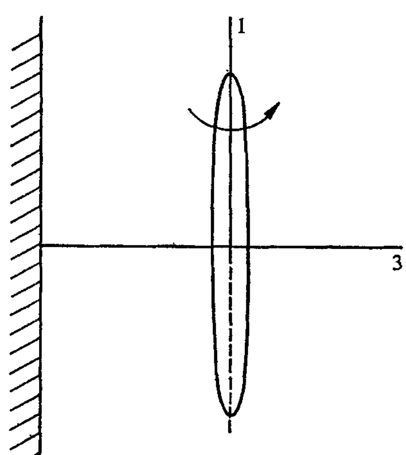

FIGURE 4. Body rotating about its symmetry axis in the neighbourhood of a plane solid wall.

with a value of

$$(U_W)_i = +Cr_2\delta_{i3} + Ck\delta_{i2}$$

on the wall W at $r_3 = -k$.

Since the wall is at a distance of order a from the body it is situated entirely within the outer region of expansion. Hence the analysis of the inner expansion given in § 3 remains unaltered. The asymptotic form of $\mathbf{u}_2$ in the outer expansion is therefore still given by (4.2). However, in dividing this velocity field into flow fields ($\mathbf{u}_2^*$, p_2^*) and ($\mathbf{u}_2'$, p_2'), it is seen that ($\mathbf{u}_2^*$, p_2^*) is now given by (cf. equation (4.6))

$$\left.\begin{aligned}(u_2^*)_i &= -\frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial R_k}[f_{ij}(\mathbf{r},\mathbf{R})]\,dR_1,\\ p_2^* &= -\frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial R_k}[g_j(\mathbf{r},\mathbf{R})]\,dR_1,\end{aligned}\right\} \tag{6.1}$$

and represents the flow field produced by a line of force doublets in the presence of the walls W. The quantities $f_{ij}(\mathbf{r},\mathbf{R})$ and $g_j(\mathbf{r},\mathbf{R})$ in the above expressions are the velocity and pressure Green's function for creeping motion flow in the presence of the walls W; i.e. $f_{ij}(\mathbf{r},\hat{\mathbf{r}})$ and $g_j(\mathbf{r},\hat{\mathbf{r}})$ are defined by the equations

$$f_{ij,kk} - g_{j,i} + \delta_{ij}\delta(\mathbf{r}-\hat{\mathbf{r}}) = 0, \quad f_{ij,i} = 0, \tag{6.2}$$

with the boundary condition

$$f_{ij} = 0 \quad \text{on} \quad W, \tag{6.3}$$

$\delta(\mathbf{r}-\hat{\mathbf{r}})$ being the Dirac delta function.

The expression for the velocity field in (6.1) may be written as

$$\begin{aligned}(u_2^*)_i = &\int_{-1}^{+1}\frac{g_{jk}(R_1)}{8\pi}\frac{\partial}{\partial r_k}\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|}+\frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right]dR_1\\ &-\int_{-1}^{+1}\frac{g_{jk}(R_1)}{8\pi}\frac{\partial}{\partial R_k}\left[f_{ij}(\mathbf{r},\mathbf{R})-\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|}-\frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right]dR_1,\end{aligned} \tag{6.4}$$

where the integrand of the second integral possesses no singularity at $\mathbf{R}=\mathbf{r}$. Now, as $\rho\to 0$, the first integral in (6.4) possesses the asymptotic expansion (4.15). Hence the expression (6.4) for $\mathbf{u}_2^*$ has the asymptotic expansion (cf. equation (4.15))

$$\begin{aligned}8\pi(u_2^*)_\rho \sim\ &\rho^{-1}[-4g_{23}\sin\theta\cos\theta-4g_{32}\sin\theta\cos\theta+2g_{22}(1-2\cos^2\theta)\\ &+2g_{33}(1-2\sin^2\theta)]+\ln\rho\left[2\frac{\partial g_{12}}{\partial r_1}\cos\theta+2\frac{\partial g_{13}}{\partial r_1}\sin\theta\right]\\ &+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(r_1-R_1)(g_{12}\cos\theta+g_{13}\sin\theta)}{|r_1-R_1|^3}dR_1\\ &+\left[\left(\frac{\partial g_{12}}{\partial r_1}\cos\theta+\frac{\partial g_{13}}{\partial r_1}\sin\theta\right)(-\ln 4-2\ln\epsilon+2)\right.\\ &\left.+\left(2\frac{\partial g_{12}}{\partial r_1}\cos\theta+2\frac{\partial g_{13}}{\partial r_1}\sin\theta\right)+8\pi(V_2^*\cos\theta+V_3^*\sin\theta)\right],\end{aligned} \tag{6.5a}$$

$$\begin{aligned}8\pi(u_2^*)_\theta \sim\ &\rho^{-1}[2g_{23}-2g_{32}]+\ln\rho\left[-2\frac{\partial g_{12}}{\partial r_1}\sin\theta+2\frac{\partial g_{13}}{\partial r_1}\cos\theta\right]\\ &+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(r_1-R_1)(-g_{12}\sin\theta+g_{13}\cos\theta)}{|r_1-R_1|^3}dR_1\\ &+\left[\left(-\frac{\partial g_{12}}{\partial r_1}\sin\theta+\frac{\partial g_{13}}{\partial r_1}\cos\theta\right)(-\ln 4-2\ln\epsilon+2)\right.\\ &\left.+8\pi(-V_2^*\sin\theta+V_3^*\cos\theta)\right],\end{aligned} \tag{6.5b}$$

$$8\pi(u_2^*)_z \sim \rho^{-1}[-4g_{12}\cos\theta - 4g_{13}\sin\theta] + \ln\rho\left[2\frac{\partial g_{22}}{\partial r_1} + 2\frac{\partial g_{33}}{\partial r_1}\right] + \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{(g_{22}+g_{33})(r_1-R_1)}{|r_1-R_1|^3}dR_1 + \left[2\left(\frac{\partial g_{23}}{\partial r_1} + \frac{\partial g_{32}}{\partial r_1}\right)\sin\theta\cos\theta + \frac{\partial g_{22}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\cos^2\theta) + \frac{\partial g_{33}}{\partial r_1}(-\ln 4 - 2\ln\epsilon + 2 + 2\sin^2\theta) + 8\pi V_1^*\right], \quad (6.5c)$$

where $\mathbf{V}^*$ is a vector which is a function of r_1 only and is given by

$$V_i^* = \frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial R_k}\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3} - f_{ij}(\mathbf{r},\mathbf{R})\right]dR_1, \quad (6.6)$$

where $\mathbf{r}$ is the vector $(r_1, 0, 0)$. Hence, following the analysis of § 4, it is seen that g_{jk} is unaltered and is still given by (4.18). The flow field $\mathbf{u}_2'$ is taken to be the flow field produced by a line of forces $f_j(R_1)$ acting at the r_1 axis in the presence of the wall W. Thus

$$(u_2')_i = \frac{1}{8\pi}\int_{-1}^{+1} f_j(R_1)[f_{ij}(\mathbf{r},\mathbf{R})]\,dR_1, \quad (6.7)$$

where $f_{ij}(\mathbf{r},\mathbf{R})$ is the Green's function defined above. Thus we write

$$(u_2')_i = \frac{1}{8\pi}\int_{-1}^{+1} f_j(R_1)\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right]dR_1 + \frac{1}{8\pi}\int_{-1}^{+1} f_j(R_1)\left[f_{ij}(\mathbf{r},\mathbf{R}) - \frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} - \frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right]dR_1, \quad (6.8)$$

the integrand of the second integral possessing no singularity for $\mathbf{r}$ lying on the r_1 axis. It may be shown that the asymptotic expansion of $\mathbf{u}_2'$ as $\rho \to 0$ is (cf. equation (4.23))

$$\left.\begin{aligned} 8\pi(u_2')_\rho &\sim 2(f_2\cos\theta + f_3\sin\theta)(\ln 2 + \ln\epsilon - \ln\rho + 1) \\ &\quad + \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{f_2\cos\theta + f_3\sin\theta}{|r_1-R_1|}dR_1 + 8\pi(W_2^*\cos\theta + W_3^*\sin\theta), \\ 8\pi(u_2')_\theta &\sim 2(-f_2\sin\theta + f_3\cos\theta)(\ln 2 + \ln\epsilon - \ln\rho) \\ &\quad + \left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{-f_2\sin\theta + f_3\cos\theta}{|r_1-R_1|}dR_1 + 8\pi(-W_2^*\sin\theta + W_3^*\cos\theta), \\ 8\pi(u_2')_z &\sim 4f_1(\ln 2 + \ln\epsilon - \ln\rho - \tfrac{1}{2}) + 2\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{f_1}{|r_1-R_1|}dR_1 + 8\pi W_1^*, \end{aligned}\right\} \quad (6.9)$$

where $\mathbf{W}^*$ is given by

$$W_i^* = \frac{1}{8\pi}\int_{-1}^{+1} f_j(R_1)\left[f_{ij}(\mathbf{r},\mathbf{R}) - \frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} - \frac{(r_i-R_i)(r_j-R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right]dR_1, \quad (6.10)$$

where $\mathbf{r}$ is the vector $(r_1, 0, 0)$. We again expand $\mathbf{f}(R_1)$ in the form (4.25) and define $\mathbf{W}_0^*, \mathbf{W}_1^*, \mathbf{W}_2^*, \ldots$ to be the vector $\mathbf{W}^*$ given by (6.10) with $\mathbf{f}(R_1)$ replaced respectively by $\mathbf{f}_0(R_1), \mathbf{f}_1(R_1), \mathbf{f}_2(R_1), \ldots$. Thus $\mathbf{W}^*$ may be expressed as

$$\mathbf{W}^* = \mathbf{W}_0^* + \frac{\mathbf{W}_1^*}{\ln\kappa} + \frac{\mathbf{W}_2^*}{(\ln\kappa)^2} + \ldots. \quad (6.11)$$

As in § 4, the value of $\mathbf{f}_0$ must be given by

$$(f_0)_1 = 0, \quad (f_0)_2 = 4\pi D\lambda\, d\lambda/dr_1, \quad (f_0)_3 = 4\pi E\lambda\, d\lambda/dr_1. \tag{6.12}$$

However the flow field $\mathbf{u}_2^* + \mathbf{u}_{20}$ has now the asymptotic expansion near $\rho = 0$ of (cf. equation (4.32))

$$\left.\begin{aligned}(u_2^*)_\rho + (u_{20})_\rho &\sim \rho^{-1}(-2A\lambda^2\cos 2\theta - 2B\lambda^2\sin 2\theta) + (2\lambda\, d\lambda/dr_1)(D\cos\theta + E\sin\theta)\\ &\quad + (V_2^* + W_2^*)\cos\theta + (V_3^* + W_3^*)\sin\theta,\\ (u_2^*)_\theta + (u_{20})_\theta &\sim \rho^{-1}(-C\lambda^2) - (V_2^* + W_2^*)\sin\theta + (V_3^* + W_3^*)\cos\theta,\\ (u_2^*)_z + (u_{20})_z &\sim \rho^{-1}(-D\lambda^2\cos\theta - E\lambda^2\sin\theta)\\ &\quad + (\lambda\, d\lambda/dr_1)(2B\sin 2\theta + 2A\cos 2\theta) + (V_1^* + W_1^*).\end{aligned}\right\} \tag{6.13}$$

Hence it may be shown that the inner flow field $\bar{\mathbf{u}}_2'$ is such that when expressed in outer variables (cf. equation (4.34))

$$\left.\begin{aligned}(\bar{u}_2')_\rho &\sim (\lambda\, d\lambda/dr_1)(D\cos\theta + E\sin\theta) + (V_2^* + W_2^*)\cos\theta\\ &\quad + (V_3^* + W_3^*)\sin\theta,\\ (\bar{u}_2')_\theta &\sim (\lambda\, d\lambda/dr_1)(-D\sin\theta + E\cos\theta) - (V_2^* + W_2^*)\sin\theta\\ &\quad + (V_3^* + W_3^*)\cos\theta,\\ (\bar{u}_2')_z &\sim (V_1^* + W_1^*).\end{aligned}\right\} \tag{6.14}$$

Thus, taking

$$\left.\begin{aligned}(\bar{u}_2')_\rho &= H\{1 - \lambda^2\bar{\rho}^{-2} - 2\ln(\bar{\rho}/\lambda)\}\cos\theta\\ &\quad + K\{1 - \lambda^2\bar{\rho}^{-2} - 2\ln(\bar{\rho}/\lambda)\}\sin\theta,\\ (\bar{u}_2')_\theta &= H\{1 - \lambda^2\bar{\rho}^{-2} + 2\ln(\bar{\rho}/\lambda)\}\sin\theta\\ &\quad + K\{-1 + \lambda^2\bar{\rho}^{-2} - 2\ln(\bar{\rho}/\lambda)\}\cos\theta,\\ (\bar{u}_2')_z &= L\ln(\bar{\rho}/\lambda),\end{aligned}\right\} \tag{6.15}$$

where H, K and L are of the form

$$\left.\begin{aligned}H &= \frac{H_1}{\ln\kappa} + \frac{H_2}{(\ln\kappa)^2} + \ldots,\\ K &= \frac{K_1}{\ln\kappa} + \frac{K_2}{(\ln\kappa)^2} + \ldots,\\ L &= \frac{L_1}{\ln\kappa} + \frac{L_2}{(\ln\kappa)^2} + \ldots,\end{aligned}\right\} \tag{6.16}$$

it may be shown that $\bar{\mathbf{u}}_2'$ expressed relative to outer variables is

$$\left.\begin{aligned}(\bar{u}_2')_\rho &= (2H_1\cos\theta + 2K_1\sin\theta) + (1/\ln\kappa)\{(2H_2 + H_1 - 2H_1\ln(\rho/\lambda))\cos\theta\\ &\quad + (2K_2 + K_1 - 2K_1\ln(\rho/\lambda))\sin\theta\} + \ldots,\\ (\bar{u}_2')_\theta &= (-2H_1\sin\theta + 2K_1\cos\theta) + (1/\ln\kappa)\{(-2H_2\\ &\quad + H_1 + 2H_1\ln(\rho/\lambda))\sin\theta + (2K_2 - K_1 - 2K_1\ln(\rho/\lambda))\cos\theta\} + \ldots,\\ (\bar{u}_2')_z &= -L_1 + (1/\ln\kappa)(-L_2 + L_1\ln(\rho/\lambda)) + \ldots.\end{aligned}\right\} \tag{6.17}$$

Hence by (6.14), it is seen that the values of H_1, K_1 and L_1 are

$$\left.\begin{aligned}H_1 &= \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1)D + V_2^* + (W_0^*)_2\},\\ K_1 &= \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1)E + V_3^* + (W_0^*)_3\},\\ L_1 &= -V_1^* - (W_0^*)_1,\end{aligned}\right\} \tag{6.18}$$

where it has been noted that $\mathbf{W}^*$ is given by the expression (6.11). The asymptotic form of $\mathbf{u}_{21}$ for $\rho \to 0$ is now given by (cf. equation (4.40))

$$\begin{aligned}8\pi(u_{21})_\rho \sim{}& \ln\rho\{-2(f_1)_2\cos\theta-2(f_1)_3\sin\theta\}+\{-2(f_1)_2\cos\theta-2(f_1)_3\sin\theta\}\\&\times(-\ln 2-\ln\epsilon-1)+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_2\cos\theta+(f_1)_3\sin\theta}{|r_1-R_1|}dR_1\\&+8\pi\{(W_1^*)_2\cos\theta+(W_1^*)_3\sin\theta\},\end{aligned}\tag{6.19a}$$

$$\begin{aligned}8\pi(u_{21})_\theta \sim{}& \ln\rho\{2(f_1)_2\sin\theta-2(f_1)_3\cos\theta\}+\{2(f_1)_2\sin\theta-2(f_1)_3\cos\theta\}(-2\ln 2-\ln\epsilon)\\&+\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{-(f_1)_2\sin\theta+(f_1)_3\cos\theta}{|r_1-R_1|}dR_1\\&+8\pi\{-(W_1^*)_2\sin\theta+(W_1^*)_3\cos\theta\},\end{aligned}\tag{6.19b}$$

$$\begin{aligned}8\pi(u_{21})_z \sim{}& -4(f_1)_1\ln\rho+4(f_1)_1(\ln 2+\ln\epsilon-\tfrac{1}{2})\\&+2\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_1}{|r_1-R_1|}dR_1+8\pi(W_1^*)_1.\end{aligned}\tag{6.19c}$$

The matching of the terms involving $\ln\rho$ gives

$$(f_1)_1=-\tfrac{1}{4}L_1.8\pi,\quad (f_1)_2=H_1.8\pi,\quad (f_1)_3=K_1.8\pi.\tag{6.20}$$

Continuing the matching process the values of H_2, K_2 and L_2 may be obtained as

$$\left.\begin{aligned}H_2&=H_1(\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon)+\tfrac{1}{2}(W_1^*)_2+\frac{1}{16\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_2}{|r_1-R_1|}dR_1,\\K_2&=K_1(\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon)+\tfrac{1}{2}(W_1^*)_3+\frac{1}{16\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_3}{|r_1-R_1|}dR_1,\\L_2&=L_1(-\tfrac{1}{2}-\ln\lambda+\ln 2+\ln\epsilon)-(W_1^*)_1-\frac{1}{4\pi}\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(f_1)_1}{|r_1-R_1|}dR_1,\end{aligned}\right\}\tag{6.21}$$

and the value of $\mathbf{f}_2$ as

$$(f_2)_1=-2\pi L_2,\quad (f_2)_2=8\pi H_2,\quad (f_2)_3=8\pi K_2.\tag{6.22}$$

Thus it is seen that the velocity field in the outer expansion is given by (5.7), where $\mathbf{U}$ is the undisturbed flow (5.8) and where $\mathbf{u}_2^*$ is the flow

$$(u_2^*)_i=-\frac{1}{8\pi}\int_{-1}^{+1}g_{jk}(R_1)\frac{\partial}{\partial R_k}[f_{ij}(\mathbf{r},\mathbf{R})]\,dR_1\tag{6.23}$$

produced by a line of force doublets in the presence of W, where $g_{jk}(R_1)$ is the matrix

$$g_{jk}(R_1)=2\pi\lambda^2\begin{pmatrix}0&0&E\\0&2A&(2B-C)\\0&(2B+C)&-2A\end{pmatrix}.\tag{6.24}$$

The remaining flow field $\kappa^2\{\mathbf{u}_{20}+(\mathbf{u}_{21}/\ln\kappa)+(\mathbf{u}_{22}/(\ln\kappa)^2)\}$ in (5.7) is given by

$$\left\{(u_{20})_i+\frac{(u_{21})_i}{\ln\kappa}+\frac{(u_{22})_i}{(\ln\kappa)^2}\right\}=\frac{1}{8\pi}\int_{-1}^{+1}\left\{(f_0)_j+\frac{(f_1)_j}{\ln\kappa}+\frac{(f_2)_j}{(\ln\kappa)^2}\right\}[f_{ij}(\mathbf{r},\mathbf{R})]\,dR_1,\tag{6.25}$$

which is a flow produced by a line of force. The quantities $(f_0)_j$, $(f_1)_j$ and $(f_2)_j$ are given by

$$f_0=4\pi\lambda\frac{d\lambda}{dr_1}\begin{pmatrix}0\\D\\E\end{pmatrix},\quad f_1=8\pi\begin{pmatrix}-\tfrac{1}{4}L_1\\H_1\\K_1\end{pmatrix},\quad f_2=8\pi\begin{pmatrix}-\tfrac{1}{4}L_2\\H_2\\K_2\end{pmatrix},\tag{6.26}$$

where L_1, H_1, K_1, L_2, H_2 and K_2 are given by

$$\left.\begin{aligned} L_1 &= -V_1^* - (W_0^*)_1, \\ H_1 &= \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1) D + V_2^* + (W_0^*)_2\}, \\ K_1 &= \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1) E + V_3^* + (W_0^*)_3\}, \end{aligned}\right\} \tag{6.27}$$

$$\left.\begin{aligned} L_2 = {} & \{V_1^* + (W_0^*)_1\}(\tfrac{1}{2} + \ln\lambda - \ln 2 - \ln\epsilon) - (W_1^*)_1 \\ & -\frac{1}{2}\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} \frac{V_1^* + (W_0^*)_1}{|r_1 - R_1|} dR_1, \\ H_2 = {} & \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1) D + V_2^* + (W_0^*)_2\}(\tfrac{1}{2} - \ln\lambda + \ln 2 + \ln\epsilon) + \tfrac{1}{2}(W_1^*)_2 \\ & +\frac{1}{4}\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} \frac{\{(\lambda\, d\lambda/dr_1) D + V_2^* + (W_0^*)_2\}}{|r_1 - R_1|} dR_1, \\ K_2 = {} & \tfrac{1}{2}\{(\lambda\, d\lambda/dr_1) E + V_3^* + (W_0^*)_3\}(\tfrac{1}{2} - \ln\lambda + \ln 2 + \ln\epsilon) + \tfrac{1}{2}(W_1^*)_3 \\ & +\frac{1}{4}\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} \frac{\{(\lambda\, d\lambda/dr_1) E + V_3^* + (W_0^*)_3\}}{|r_1 - R_1|} dR_1, \end{aligned}\right\} \tag{6.28}$$

the values of the vectors $\mathbf{V}^*$, $\mathbf{W}_0^*$ and $\mathbf{W}_1^*$ being given by

$$\left.\begin{aligned} V_i^* &= \frac{1}{8\pi}\int_{-1}^{+1} g_{jk}(R_1)\frac{\partial}{\partial R_k}\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} + \frac{(r_i - R_i)(r_j - R_j)}{|\mathbf{r}-\mathbf{R}|^3} - f_{ij}(\mathbf{r},\mathbf{R})\right] dR_1, \\ (W_0^*)_i &= \frac{1}{8\pi}\int_{-1}^{+1} (f_0)_j\left[f_{ij}(\mathbf{r},\mathbf{R}) - \frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} - \frac{(r_i - R_i)(r_j - R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1, \\ (W_1^*)_i &= \frac{1}{8\pi}\int_{-1}^{+1} (f_1)_j\left[f_{ij}(\mathbf{r},\mathbf{R}) - \frac{\delta_{ij}}{|\mathbf{r}-\mathbf{R}|} - \frac{(r_i - R_i)(r_j - R_j)}{|\mathbf{r}-\mathbf{R}|^3}\right] dR_1, \end{aligned}\right\} \tag{6.29}$$

the vector $\mathbf{r}$ being put equal to $(r_1, 0, 0)$ in each of these integrals.

The forces acting on the body are therefore equivalent to a force $\mathscr{F}$ and a couple $\mathscr{G}$ per unit length acting on it, where

$$\mathscr{F}/8\pi = \kappa^2 \begin{pmatrix} \dfrac{1}{4}\dfrac{L_1}{\ln\kappa} + \dfrac{1}{4}\dfrac{L_2}{(\ln\kappa)^2} \\ -\tfrac{1}{2}D\lambda\dfrac{d\lambda}{dr_1} - \dfrac{H_1}{\ln\kappa} - \dfrac{H_2}{(\ln\kappa)^2} \\ -\tfrac{1}{2}E\lambda\dfrac{d\lambda}{dr_1} - \dfrac{K_1}{\ln\kappa} - \dfrac{K_2}{(\ln\kappa)^2} \end{pmatrix},$$

$$\mathscr{G}/8\pi = \tfrac{1}{4}\kappa^2\lambda^2 \begin{pmatrix} 2C \\ E \\ -D \end{pmatrix}. \tag{6.30}$$

The total force $\mathbf{F}$ and torque $\mathbf{G}$ acting on the body about the origin are given by

$$\mathbf{F} = \int_{-1}^{+1} \mathscr{F}\, dr_1, \quad \mathbf{G} = \int_{-1}^{+1} (\mathscr{G} + \mathbf{r}\times\mathscr{F})\, dr_1. \tag{6.31}$$

These equations give, for the force **F**,

$$\left.\begin{aligned} F_1/8\pi &= \frac{1}{4}\frac{\kappa^2}{\ln\kappa}\int_{-1}^{+1} L_1\,dr_1 + \frac{1}{4}\frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} L_2\,dr_2,\\ F_2/8\pi &= -\frac{\kappa^2}{\ln\kappa}\int_{-1}^{+1} H_1\,dr_1 - \frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} H_2\,dr_2,\\ F_3/8\pi &= -\frac{\kappa^2}{\ln\kappa}\int_{-1}^{+1} K_1\,dr_1 - \frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} K_2\,dr_2, \end{aligned}\right\} \tag{6.32}$$

and, for the torque **G**,

$$\left.\begin{aligned} G_1/8\pi &= \tfrac{1}{2}\kappa^2 C\int_{-1}^{+1} \lambda^2\,dr_1,\\ G_2/8\pi &= +\frac{\kappa^2}{\ln\kappa}\int_{-1}^{+1} r_1 K_1\,dr_1 + \frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} r_1 K_2\,dr_1,\\ G_3/8\pi &= -\frac{\kappa^2}{\ln\kappa}\int_{-1}^{+1} r_1 H_1\,dr_1 - \frac{\kappa^2}{(\ln\kappa)^2}\int_{-1}^{+1} r_1 H_2\,dr_1, \end{aligned}\right\} \tag{6.33}$$

where L_1, H_1, K_1, L_2, H_2 and K_2 are given above by (6.27) and (6.28).

When there is no wall present, then $\mathbf{V}^* = \mathbf{W}_0^* = \mathbf{W}_1^* = 0$ and (6.32) and (6.33) reduce to the results given at the end of § 5.

7. Resistance to axial rotation

In this and the next two sections, the results given in § 5 for the force and torque on a long slender body will be used to examine special cases.

Consider first a long slender body which is rotating with a dimensionless angular velocity of unity about the r_1 axis. The fluid in which such a body is immersed is assumed to be unbounded and at rest at infinity. The disturbance produced by the body and the forces acting on the body are the same as would be produced by taking the body at rest in a fluid undergoing a motion $\mathbf{U}(\mathbf{r})$ given by

$$U_\rho = 0, \quad U_\theta = -\rho, \quad U_z = 0. \tag{7.1}$$

Thus in (2.7) one has

$$A = B = D = E = 0, \quad C = -1. \tag{7.2}$$

Hence by equations (5.18), (5.22) and (5.23), it is seen that the force **F** and torque **G** acting on the body are given by

$$\mathbf{F} = 0 \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} G_1 \\ 0 \\ 0 \end{pmatrix}, \tag{7.3}$$

where

$$G_1 = -4\pi\kappa^2\int_{-1}^{+1} \lambda^2\,dr_1. \tag{7.4}$$

Thus in *dimensional* variables it is seen that the couple acting on a body rotating with angular velocity ω is

$$\begin{aligned} G_1 &= -4\pi\mu a b^2\omega\int_{-1}^{+1} \lambda^2\,dr_1\\ &= -4\mu\omega V, \end{aligned} \tag{7.5}$$

where V is the volume of the body and μ the viscosity of the fluid.

Although the results given in § 5 were derived for bodies with 'sharp ends' for which $\lambda(r_1)$ is a continuous function satisfying $\lambda(-1) = \lambda(+1) = 0$, it may be seen that the present result (7.5) for rotational resistance is valid for blunt-ended bodies (such as a circular cylinder of finite length) since the effect of the ends is to alter the value of the couple by an amount of order $(\mu b^3 \omega)$ which is much smaller than that given by (7.5) for the total couple. Thus one may expect this equation (7.5) to be valid for bodies for which $\lambda(r_1)$ is piecewise continuous with a finite number of discontinuities.

8. Force and torque on body in shear

Consider an undisturbed shear flow $\mathbf{U}(\mathbf{r})$ given by

$$U_\rho = U_\theta = 0, \quad U_z = \rho \sin\theta, \tag{8.1}$$

which is the flow field (2.7) with

$$A = B = C = D = 0, \quad E = 1. \tag{8.2}$$

In this flow a long slender body is considered held at rest with its axis in the r_1 direction. The hydrodynamic force $\mathbf{F}$ and torque $\mathbf{G}$ acting on the body are now calculated using equations (5.22) and (5.23) for the cases in which the shape of the body is (i) a double cone and (ii) an ellipsoid of revolution.

(i) *A double cone*

If the body shape is a double cone (see figure 5(*a*)) given by

$$\begin{aligned} \lambda(r_1) &= 1 + (r_1/\alpha) \quad \text{for} \quad -\alpha \leqslant r_1 \leqslant 0, \\ &= 1 - r_1 \qquad \text{for} \quad 0 \leqslant r_1 \leqslant 1, \end{aligned} \tag{8.3}$$

then it may readily be shown that if $-\alpha \leqslant r_1 \leqslant 0$

$$\begin{aligned} \left\{\int_{-\alpha}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} \frac{(\lambda\, d\lambda/dR_1)\, dR_1}{|r_1 - R_1|} = &-2\alpha^{-1}(1+\alpha^{-1}r_1)\ln\epsilon \\ &+ (1+\alpha^{-1}+\alpha^{-2}r_1 - r_1)\ln(-r_1) \\ &+ \alpha^{-1}(1+\alpha^{-1}r_1)\ln(r_1+\alpha) + (r_1-1)\ln(1-r_1) \\ &+ (1-\alpha^{-1}-2\alpha^{-2}r_1), \end{aligned} \tag{8.4a}$$

and if $0 \leqslant r_1 \leqslant 1$

$$\begin{aligned} \left\{\int_{-\alpha}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\} \frac{(\lambda\, d\lambda/dR_1)\, dR_1}{|r_1 - R_1|} = &-2(r_1-1)\ln\epsilon + (-1-\alpha^{-1}-\alpha^{-2}r_1+r_1)\ln r_1 \\ &+ \alpha^{-1}(1+\alpha^{-1}r_1)\ln(r_1+\alpha) + (r_1-1)\ln(1-r_1) \\ &+ (1-\alpha^{-1}-2r_1). \end{aligned} \tag{8.4b}$$

Substituting these values into (5.22) and making use of (8.2), one obtains the value of the force $\mathbf{F}$ on the body as $\quad F_1 = F_2 = 0,$

$$F_3/8\pi = -[\kappa^2/(\ln\kappa)^2]\,.\,\tfrac{1}{4}\{\tfrac{1}{2}(1+\alpha)^2\ln\alpha + \tfrac{1}{2}(\alpha^{-2}-1)(1+\alpha)^2\ln(1+\alpha) + \tfrac{1}{2}(\alpha-\alpha^{-1})\}. \tag{8.5}$$

Similarly, the value of the torque $\mathbf{G}$ on the body about the origin is obtained by substituting the values (8.4) into (5.23) and by making use of (8.2) and (8.3). Thus $\quad G_1 = G_3 = 0,$

$$G_2/8\pi = -[\kappa^2/\ln\kappa].\tfrac{1}{12}(1+\alpha)-[\kappa^2/(\ln\kappa)^2].\tfrac{1}{24}[(2\alpha^3+3\alpha^2+\alpha)\ln\alpha + (-2\alpha^3-3\alpha^2+\alpha+1-3\alpha^{-1}-2\alpha^{-2})\ln(1+\alpha)+2\ln 2(1+\alpha) + (2\alpha^2+\alpha+1+2\alpha^{-1})]. \tag{8.6}$$

Thus, if the magnitude of the shear is γ, the *dimensional* force and torque may be written as

$$F_1 = F_2 = 0,$$

$$F_3 = -[\mu b^2\gamma/(\ln a/b)^2]\pi\{(1+\alpha)^2\ln\alpha+(\alpha^{-2}-1)(1+\alpha)^2\ln(1+\alpha)+(\alpha-\alpha^{-1})\}, \tag{8.7}$$

and
$$G_1 = G_3 = 0, \quad G_2 = \tfrac{8}{3}\pi\mu ab^2\gamma(1+\alpha)/[\ln(a/b)+C], \tag{8.8}$$

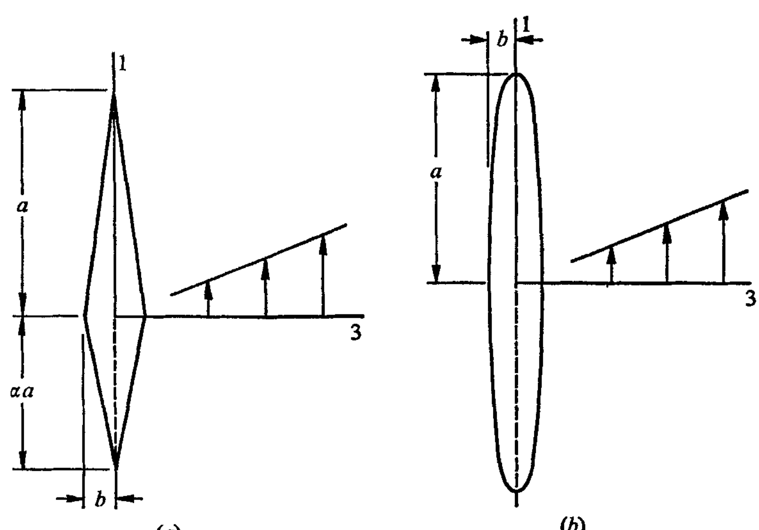

FIGURE 5. (*a*) Double cone in shear flow. (*b*) Ellipsoid of revolution in shear flow.

where C is a constant given by

$$C = \tfrac{1}{2}\{(2\alpha^2+\alpha)\ln\alpha+(-2\alpha^2-\alpha+2-\alpha^{-1}-2\alpha^{-2})\ln(1+\alpha) + 2\ln 2+(2\alpha-1+2\alpha^{-1})\}. \tag{8.9}$$

(ii) *An ellipsoid*

Consider a body of ellipsoidal shape (figure 5(*b*)) given by

$$\lambda(r_1) = \pm(1-r_1^2)^{\frac{1}{2}} \quad (-1 \leqslant r_1 \leqslant +1). \tag{8.10}$$

By direct substitution it may then be shown that

$$\left\{\int_{-1}^{r_1-\epsilon}+\int_{r_1+\epsilon}^{+1}\right\}\frac{(\lambda\, d\lambda/dR_1)\,dR_1}{|r_1-R_1|} = 2r_1\ln\epsilon+2r_1-r_1\ln(1-r_1^2) \tag{8.11}$$

for $-1 \leqslant r_1 \leqslant +1$. The substitution of this expression into (5.22) gives the total force $\mathbf{F}$ on the body as

$$\mathbf{F} = 0 \tag{8.12}$$

whilst substitution into (5.23) gives the total torque $\mathbf{G}$ about the origin as

$$G_1 = G_3 = 0,$$

$$G_2/8\pi = -\tfrac{1}{3}[\kappa^2/\ln\kappa]-(-\tfrac{1}{6}+\tfrac{1}{3}\ln 2)[\kappa^2/(\ln\kappa)^2]. \tag{8.13}$$

The dimensional force and torque on the body may therefore be written in the form

$$\mathbf{F} = 0 \tag{8.14}$$

and

$$G_1 = G_3 = 0,$$
$$G_2 = \tfrac{8}{3}\pi\mu ab^2\gamma/[(\ln a/b) + (\ln 2 - \tfrac{1}{2})]. \tag{8.15}$$

It should be noted that, for the body whose shape is a double cone, there is in general a total force acting on it across the flow in the r_3 direction (see equation (8.7)). For example, if $\alpha = \frac{1}{2}$, then the force $\mathbf{F}$ on the double cone is

$$F_1 = F_2 = 0,$$
$$F_3 = +1{\cdot}013\,\mu b^2\gamma/(\ln a/b)^2. \tag{8.16}$$

However, for the symmetric double cone ($\alpha = 1$) and for the ellipsoid there is zero force on the body. In general it may be shown from (5.22) that there is zero total force on any such axisymmetric body which possesses fore–aft symmetry, a result which may also readily be demonstrated by considerations of symmetry alone.

The general results (5.22) and (5.23) for the shear flow (8.1) always give the dimensional force and couple on a body in the form

$$F_1 = F_2 = 0,$$
$$F_3 = 2\pi K_1\mu b^2\gamma/(\ln a/b)^2, \tag{8.17}$$

and

$$G_1 = G_3 = 0,$$
$$G_2 = 2\pi\mu ab^2\gamma\left\{\frac{K_2}{\ln a/b} + \frac{K_3}{(\ln a/b)^2}\right\}, \tag{8.18}$$

where K_1, K_2 and K_3 are constants given by

$$K_1 = -\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{(\lambda\, d\lambda/dR_1)\, dR_1}{|r_1 - R_1|}\right] dr_1,$$

$$K_2 = +\int_{-1}^{+1}\lambda^2\, dr_1,$$

$$K_3 = -(1 + \ln 2 + \ln\epsilon)\int_{-1}^{+1}\lambda^2\, dr_1 + \int_{-1}^{+1}\lambda^2\ln\lambda\, dr_1$$
$$+\int_{-1}^{+1}\left[\left\{\int_{-1}^{r_1-\epsilon} + \int_{r_1+\epsilon}^{+1}\right\}\frac{(\lambda\, d\lambda/dR_1)\, dR_1}{|r_1 - R_1|}\right] r_1 dr_1. \tag{8.19}$$

These results are valid for bodies for which $\lambda(r_1)$ is a continuous function taking the value zero at the ends of the body. They are not valid for blunt-ended bodies (such as a circular cylinder of finite length) since the effect of the ends is to give a force on the body of order $(\mu b^2\gamma)$ and a torque on the body about the origin of order $(\mu ab^2\gamma)$ which are *larger* than the values given by (8.18). Thus for blunt-ended bodies with fore–aft symmetry in shear flow given by (8.1) [for which $\mathbf{F} = 0$] one would expect the total torque on the body to be given by

$$\left.\begin{aligned} G_1 &= G_3 = 0, \\ G_2 &= L\mu ab^2\gamma, \end{aligned}\right\} \tag{8.20}$$

where L is a constant which depends critically upon the shape of the blunt ends of the body.

9. Equivalent axis ratio

The motion of an axially symmetric body with fore–aft symmetry in shear flow is given by (1.2) and (1.3). This motion is determined by the initial conditions and also by the value of a single constant r_e which is dependent only upon the body shape. It was shown in § 1 that

$$r_e = (G'/G'')^{\frac{1}{2}}, \tag{9.1}$$

where G' and G'' are the couples exerted on the body when it is held at rest in a shear flow with its axis respectively across and in the direction of the flow. For sharp-ended bodies for which $\lambda(r_1)$ is continuous and takes a value of zero at the body ends, it is seen from § 8 that

$$G'' = 2\pi\mu ab^2\gamma\left\{\frac{K_2}{\ln a/b}+\frac{K_3}{(\ln a/b)^2}\right\}, \tag{9.2}$$

where K_2 and K_3 are given by (8.19). Also from the general results given by Cox (1970) for a long slender body it may readily be shown that the value of G' is

$$G' = \mu a^3\gamma\left(\frac{8\pi}{3}\right)\left\{\frac{1}{\ln a/b}+\frac{K_4}{(\ln a/b)^2}\right\}, \tag{9.3}$$

where K_4 is a constant given by

$$K_4 = -(\ln 2-\tfrac{1}{2})-\frac{3}{4}\int_{-1}^{+1} r_1^2 \ln\left(\frac{1-r_1^2}{\lambda^2}\right)dr_1. \tag{9.4}$$

Substituting the values of G'' and G' given by (9.2) and (9.3) in (9.1) and expanding in powers of $1/(\ln a/b)$, one obtains

$$r_e\Big/\frac{a}{b} = p\left\{1+\frac{q}{\ln a/b}\right\}, \tag{9.5}$$

where p and q are constants given by

$$p = \left(\frac{4}{3K_2}\right)^{\frac{1}{2}}, \quad q = \frac{1}{2}\left(K_4-\frac{K_3}{K_2}\right), \tag{9.6}$$

the values of K_2, K_3 and K_4 being given by (8.19) and (9.4). It is to be noted that, since the volume V of the body is given by

$$V = \pi ab^2\int_{-1}^{+1}\lambda^2\,dr_1 = \pi ab^2K_2, \tag{9.7}$$

the above expression (9.6) for the constant p may be written as

$$p = (\tfrac{4}{3}\pi ab^2/V)^{\frac{1}{2}}. \tag{9.8}$$

The ratio of the equivalent axis ratio to the true axis ratio (i.e. $r_e/(a/b)$) is seen by (9.5) to tend to a constant value p as $a/b\to\infty$ for the sharp-ended bodies considered here (i.e. bodies with $\lambda(r_1)$ continuous and $\lambda(-1) = \lambda(+1) = 0$). Whether this ratio increases or decreases to this limiting value depends upon the sign of the constant q.

As an example consider the double cone described by (8.3) with $\alpha = 1$ so that it possesses fore–aft symmetry. Then by comparing equations (9.2) and (8.6) it is seen that for this case the values of K_2 and K_3 are

$$K_2 = \tfrac{2}{3}, \quad K_3 = \tfrac{2}{3}\ln 2 - 1. \tag{9.9}$$

Also the value of K_4 given by (9.4) is for the present case

$$K_4 = -2\ln 2. \tag{9.10}$$

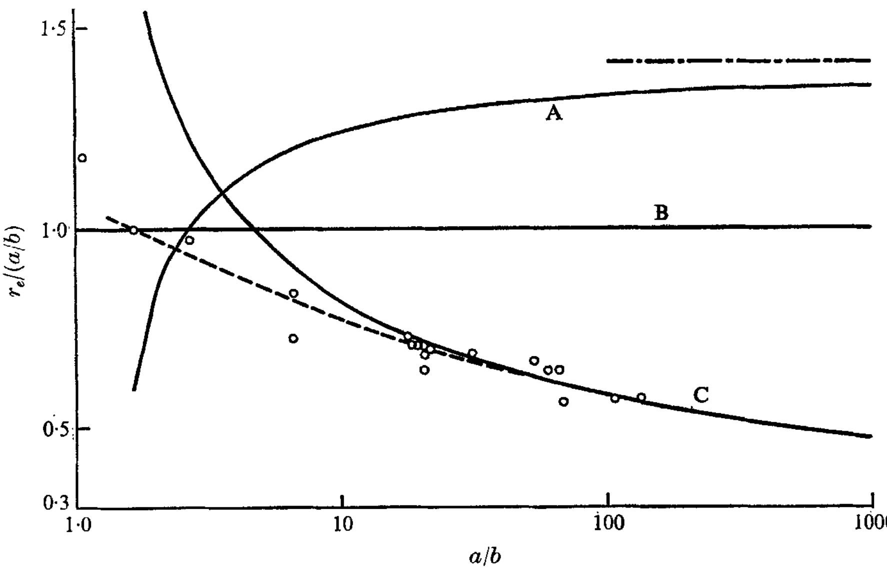

FIGURE 6. The ratio of equivalent axis ratio (r_e) to true axis ratio (a/b) as a function of true axis ratio for (A) a double cone, (B) an ellipsoid, and (C) a cylinder of finite length. The open circles represent the experimental values for a cylinder of finite length obtained by Anczurowski & Mason (1968). The line —·——·— represents the asymptotic value for large values of a/b for a double cone (case A).

Hence $$p = 2^{\frac{1}{2}}, \quad q = \tfrac{3}{4}(1-2\ln 2), \tag{9.11}$$

so that the expression (9.5) for the equivalent axis ratio r_e becomes

$$r_e\Big/\left(\frac{a}{b}\right) = 1{\cdot}414 - \frac{0{\cdot}4097}{\ln a/b}. \tag{9.12}$$

Similarly, for an ellipsoid given by

$$\lambda(r_1) = \pm(1-r_1^2)^{\frac{1}{2}},$$

the values of K_2, K_3 and K_4 are

$$K_2 = \tfrac{4}{3}, \quad K_3 = \tfrac{2}{3} - \tfrac{4}{3}\ln 2, \quad K_4 = \tfrac{1}{2} - \ln 2, \tag{9.13}$$

so that $$p = 1 \quad \text{and} \quad q = 0. \tag{9.14}$$

Therefore $$r_e/(a/b) = 1. \tag{9.15}$$

This result also follows immediately from the definition of the equivalent axis ratio r_e (see § 1). The values of $r_e/(a/b)$ in terms of (a/b) derived from (9.12) and (9.15) for the double cone and ellipsoid are given graphically in figure 6. In § 8 it was suggested that for blunt-ended bodies (or more generally bodies for which $\lambda(r_1)$ is piecewise continuous with a finite number of discontinuities in $-1 \leqslant r_1 \leqslant 1$) such as a cylinder of finite length the value of G'' is no longer given by (9.2) but is instead given by

$$G'' = L\mu a b^2 \gamma, \tag{9.16}$$

where L is a constant (see equation (8.20)). Since G' is still given by (9.3) it follows that the value of the equivalent axis ratio r_e is given by

$$r_e \bigg/ \left(\frac{a}{b}\right) = \left(\frac{8\pi}{3L}\right)^{\frac{1}{2}} (\ln a/b)^{-\frac{1}{2}}. \tag{9.17}$$

Experimental values of $r_e/(a/b)$ for different (a/b) for a finite circular cylinder are given by Anczurowski & Mason (1968). These are shown in figure 6 and are compared with the values given by (9.17) with the constant L put equal to 5·45. This value seems to give the best agreement between theory and experiment, (9.17) then having the form

$$r_e/(a/b) = 1{\cdot}24 (\ln a/b)^{-\frac{1}{2}}. \tag{9.18}$$

The equation (9.16) for the couple G'' then becomes

$$G'' = 5{\cdot}45 \mu a b^2 \gamma, \tag{9.19}$$

so that when the cylinder is aligned with the flow its ends have a force acting on them (see § 8) across the flow of magnitude $2{\cdot}72\mu b^2\gamma$.

This work was supported by the National Heart Institute of the United States Public Health Service (Grant HE-05911).

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